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TRACTS

ON

MATHEMATICAL

AND

PHILOSOPHICAL SUBJECTS;

COMPRISING,

AMONG NUMEROUS IMPORTANT ARTICLES,

THE THEORY OF BRIDGES,

WITH SEVERAL PLANS OF RECENT IMPROVEMENT.

ALSO

THE RESULTS OF NUMEROUS EXPERIMENTS ON

THE FORCE OF GUNPOWDER,

WITH APPLICATIONS TO

THE MODERN PRACTICE OF ARTILLERY.

IN THREE VOLUMES.

BY CHARLES HUTTON, LL.D. AND F.R.S. &c.

Late Professor of Mathematics in the Royal Military Academy, Woolwich.

VOL. II.

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CONTENTS TO VOL. II.

TRACT XXVI.

THE

MEAN DENSITY OF THE EARTH.

BEING AN

ACCOUNT OF THE CALCULATIONS MADE FROM THE SURVEY AND MEASURES TAKEN AT MOUNT SHICHALLIN*, IN ORDER TO ASCERTAIN THE MEAN DENSITY OF THE EARTH. IMPROVED FROM THE PHILOSOPHICAL TRANSACTIONS, VOL. 68, FOR THE YEAR 1778.

THIS very curious and elaborate paper, contains a full account of all the measurements and calculations, drawn up at the request of the Royal Society, concerning a problem of the very first importance in physics, viz, to determine the mean density and mass of the whole body of the earth; and exhibiting the results of the whole, from the very novel modes of computation here employed.

The survey from which these calculations have been made, was taken at and about the hill Shichallin in Perthshire, in the years 1774, 1775, and 1776, by the direction, and partly under the inspection, of the Rev. Nevil Maskelyne, D. D. F. R. S. and astronomer royal, by whom the manner of making the survey was pretty fully explained in the Philosophical Transactions for 1775. I have therefore only to give an ac-

* Concerning this place, (called Thichallin in the Earse or Gaelic tongue, but pronounced in English Shehallien, and signifying the maiden's breast), in the parish of Fortingal, see Sir John Sinclair's Statistical Account of Scotland.

count of the measures of the lines and angles, and of the calculations which I have raised from them, with all possible care and faithfulness, for the purpose of determining the measure of the ratio of the mean density of the earth, to that of water, or any other known matter.

These calculations were naturally and unavoidably long and tedious; and the more so as the business was in a manner quite new, which laid me under the necessity of inventing and describing such modes of computation as should be proper to be applied, in so important and delicate a business. Having, at length, with close and unwearied application for a considerable time completed all the calculations, the following sheets contain an account of those operations, with the results arising from them; and are accompanied with such drawings as are necessary to illustrate the descriptions. They exhibit also a synopsis of the measures which were taken of the lines and angles; from which any person may at any time satisfy himself of the truth of the computations that have been made, and are here described. These measures are here immediately subjoined, before proceeding to describe the computations made from them.

A synopsis of the horizontal and vertical angles that were observed at the principal points in making the survey about Shichallin.

In the first column are contained the names of the horizontal angles, the measure of which, in degrees and minutes, are in the second column; and the vertical angles are in the third column; in which it is to be observed, that the letter denoting the object is placed before the degrees and minutes, and E or D after them, to show that they are in elevation or depression respectively. The mark . , placed to the measure of any angle, denotes that it is the mean of the two observations made with the instrument turned different ways; namely, after the first observation, reversing it to make the second. Also the mean height of the theodolite is set down to each station. In the vertical angles, the bottom of the object is

understood, unless where the word top is annexed to it; and sometimes the height of the pole is added, in feet and inches. The several letters A, B, C, &c, may be seen in the plans of the survey annexed, either in plate 2 or 4.

At A	Theodolite = 4 ft. 10 in.	
	° ′	° ′
DAB	31 15	
DAN	77 30	N 12 22 E
DAO	102 36½	O 0 58 E
DAR	134 31½	
NAC	83 13½	N 12 20 E
NAO	25 4	
OAS	27 24	S 5 25 D
OAR	31 55	R 6 31 D
At B		
DBN	81 8	
DBO	101 41	
DBA	139 59	
At C	Theodolite = 4 ft. 7½ in.	
ACD	126 6	
ACF	93 34	
ACG	92 15	
ACH	85 31½	
ACN	49 10	N 10 4 E
ACB	8 11	
At D	Theodolite = 4 ft. 8½ in.	
ADC	48 7	
ADB	8 45	
ADN	49 14	N 10 40 E
ADW	83 56½	N 10 39 E
ADH	88 56	A 1 15 E
ADG	95 5	G 3 17 D
ADF	96 3	F 3 34 D
At F	Theodolite = 4 ft. 8½ in.	
GFW	17 27	W 0 58 D
GFK	76 7¼	K 9 50 E
GFN	103 57½	N 17 42½ E
GFD	172 54	D 3 37 E
HFG	10 9	G 1 36 D
HFP	60 41	P 5 9 E
HFK	65 58	K 9 41 E
HFN	93 49	N 17 35½ E
At G	Theodolite = 4 ft. 6½ in.	
DGF	6 8	
NGD	59 41	
NGF	65 49	
NGH	101 9	
HGF	166 57	
	° ′	° ′
HGD	160 49	
HGX	71 11½	
HGW	4 32	
PGF	95 56	F 1 15 E
PGN	30 6½	N 16 54 E
		top of Cairn.
PGK	0 9¼	K 10 15 E
PGL	62 57	L 0 32 D
PGW	63 20	W 0 54 D
PGH	71 2	H 3 0 D
		P 5 46 E
		top of ball.
At H	Theodolite = 4 ft. 5½ in.	
GHF	2 54	G 2 59 E
GHD	13 2	
NHG	49 44	N 14 42 E
NHK	29 17	
NHW	107 43	N 14 39 E top
NHW	107 41	
KHD	65 59	K 12 6 E top
KHG	79 1	
KHW	78 25	W 3 2 E
WHP	96 14	P 7 39 E top
WH*p*	149 2	*p* 0 22 E
WHG	157 25½	G 2 55 E
At W	Theodolite = 4 ft. 5½ in.	
LWK	107 27½	L 0 25½ E
LWP	130 40	P 5 43 E top
LWP	130 41	
LWN	133 48½	N 12 36½ E
LWF	175 2	K 11 30 E
LWQ	4 41	
LWG	178 21	
LWH	193 13	H 3 14 D
At L	Theodolite = 4 ft. 9 in.	
GLP	37 1	P 4 24½ E
GLY	139 52½ .	Y 2 35½ D
WLG	1 18	W 0 . 40½ D
	1 17¾ .	G 0 . 31 E
WLP	38 12	P 4 . 22½ E
	38 12½ .	
WLN	38 58	N 10 . 56½ E
	38 58½ .	
WLK	60 33	K 10 . 24½ E
	60 34 .	

	° ′	° ′
WLT	115 38 .	T 0 . 30 D
WLV	126 58½ .	V 0 . 35½ D
WLY	140 50½ .	Y 2 . 35 D
WLF	3 58½ .	F 0 . 38½ E
WLE	62 12 .	E 10 . 16½ E
WL*t*	38 4½ .	*t* 4 . 34½ E
At Y	Theodolite = 4 ft. 8 in.	
TYK	69 23¼ .	K 10 . 29 E
TYN	76 56½ .	N 9 . 37¼ E
TYL	124 47 .	L 2 . 35 E
TYG	103 24¾ .	G 1 . 36½ E
TYZ	19 21¾ .	Z 3 . 3½ E
TYV	13 29½ .	V 3 . 26½ E
VYT	13 28 .	T 2 . 4½ E
VYZ	32 52 .	Z 3 . 3½ E
VYE	81 49½ .	E 10 . 33 E
VYK	82 53 .	K 10 . 29 E
VYN	90 24 .	N 9 . 36½ E
VYG	116 52 .	G 1 . 35½ E
VYH	126 57 .	H 1 . 0 E
VYW	130 33 .	W 2 . 2 E
VYL	138 14 .	L 2 . 33½ E
At V	Theodolite = 4 ft. 11 in.	
TVZ	17 25¼ .	Z 2 . 28½ E
	17 25 .	2 . 26
TVU	34 24½ .	U 2 . 47½ E
TVE	70 43 .	E 9 . 10 E
TVK	71 23½ .	K 9 . 5½ E
	71 24 .	9 . 5½
TVW	112 32½ .	W 0 . 25½ E
TVL	119 57 .	L 0 . 34 E
TVY	147 51½ .	Y 3 . 31½ D
	147 50 .	3 . 28½ D
		T 0 6 E
At T	Theodolite = 4 ft. 10 in.	
YTV	18 41½ .	V 0 . 8¾ D
YTL	30 1 .	L 0 . 27¼ E
YTU	116 18¼ .	U 3 . 31 E
YTZ	133 31½ .	Z 3 . 49½ E
		Y 2 10 D
VTY	18 41 .	Y 2 . 6 D
VTL	48 43 .	L 0 . 30 E
VTE	94 2 .	E 9 . 42 E
VTU	135 1½ .	U 3 . 31½ E
VTZ	152 14½ .	Z 3 . 50 E
		V 0 7 D
At Z	Theodolite = 4 ft. 6½ in.	
UZR	10 53½ .	R 0 . 50 D
UZA	16 5 .	A 1 . 58¾ E
UZX	16 18 .	X 1 . 52 D
UZO	33 18¾ .	O 3 . 58 E
UZY	117 21 .	Y 3 . 8¼ D
UZV	134 4½ .	V 2 . 31½ D
UZT	144 27½ .	T 3 . 36¾ D
		U 2 . 45¼ E
At U	Theodolite = 4 ft. 10 in.	
ZUT	18 20¾ .	T 3 . 32¼ D
ZUV	28 54½ .	V 2 . 51 D
ZUO	124 42 .	O 4 . 1½ E
ZUN	125 9½ .	N 10 . 13½ E
ZUX	128 2¾ .	X 10 . 55½ D
ZUA	157 55 .	A 1 . 36¾ E
ZUR	160 37½ .	R 3 . 38 D
		Z 2 . 51 D
At X	Theodolite = 4 ft. 7 in.	
OXA	40 59½ .	A 4 . 15 E
OXS	53 20 .	S 0 . 44¼ E
OXZ	139 22½ .	U 10 . 49¼ E
OXU	174 59 .	Z 1 . 48¾ E
		O 11 . 13½ E
At S	Theodolite = 4 ft. 9 in.	
RSA	18 27	A 5 24 E
RSA	18 28	A 5 20½ E
RSN	72 51	N 18 40 E
RSO	90 58	O 13 19½ E
RSX	171 1	X 0 55½ D
		R 0 24½ E
At R the E. end of the S. base.	Theodolite = 4 ft. 6 in.	
KRN	41 6½	
B″RA	20 36¼ .	A 6 . 27½ E
B″RN.	84 32¾ .	N 19 . 17½ E
B″RO	111 46¼ .	O 11 . 57¼ E
B″Rα'	100 14½ .	α' 20 . 2 E
B″Rβ'	121 27¾ .	β' 15 . 11¼ E
B″RS	177 39	
B″RZ	186 53½ .	Z 0 . 46 E
its sup.	173 6½ .	O 12 29 E
At B″ the W. end of the S. base.	Theodolite = 4 ft. 10 in.	
RB″S	0 55½	R 0 37 D
RB″β'	36 8¼ .	β' 10 . 24 E
RB″O	37 27 .	O 7 . 35¼ E
RB″α'	50 49½ .	α' 15 . 48 E
RB″N	65 46¾ .	N 17 . 33 E
RB″A	139 2¼ .	A 11 . 22¾ E

At O the S. observatory.	Theodolite = 4 ft. 7 in.	
	° ′	° ′
XOS	35 39 .	S 13 . 25½ D
XOM	42 12¾ .	M 13 . 12½ D
XOR	58 49¾ .	R 12 . 4 D
XOB″	89 35 .	B″ 7 . 39½ D
XOA	115 43½ .	A 0 . 59 D
XOU	1 39 .	U 4 . 4½ D
XOZ	23 38½ .	Z 4 . 0 D
		X 11 . 18 D
MOR	16 36½ .	R 12 . 3½ D
MOB″	47 23 .	B″ 7 . 39½ D
MOA	73 32½ .	A 1 . 0½ D
MOB	81 22 .	B 1 . 1½ D top
MON	134 39	M 13 16
NOK	95 1	
MOα′	156 34 .	α′ 33 . 47½ E top
MOβ′	125 45 .	β′ 21 . 29½ E.
MOZ	65 51½ .	Z 4 . 0 D
MOU	43 51½ .	U 4 . 3½ D
MOX	42 13½ .	X 11 . 19 D
MOS	6 34½ .	S 13 . 22½ D

M is in the meridian passing through O, and is a little South of the intersection of that meridian and the line RS.

At P the N. observatory.	Theodolite = 4 ft. 10 in.	
NPK	77 18¾	
GPK	179 35¼ .	G 5 39¾ D
GPH	47 45¼ .	H 7 . 32½ D
GPW	69 0¾ .	W 5 . 37 D
GPL	80 9¾ .	L 4 . 27¾ D
GPF′	20 54 .	F′ 5 . 6 D
GPm	38 18¼ .	m 12 . 20 D
γ′Pδ′	45 39	by reduction
γ′PK	2 55¾	by reduction

γ′ from P bears 30° 41′⅗ E. of South.
δ′ from P bears 14 57 W. of South.
m is Mr. Mason's mark.

At N the W. cairn.	Theodolite = 4 ft. 11 in.	
KNO	40 5¾	by reduction
KNR	74 48¾	by reduction
KNP	50 47¾	
KNδ′	9 37½	
KNγ′	2 11⅗	
KNA	133 53 .	A 12 . 22½ D
KNB	146 15½ .	B 11 . 17 D top
KNC	178 30½ .	C 10 . 4 D
KND	172 51 .	D 10 . 39 D top
KNn	144 37½ .	n 4 . 30½ D
KNG	98 23 .	G 16 . 55 D top Pole 4 ft. 4 in.
KNH	69 13¼	H 14 38 D top Pole 6 ft. 5 in.
KNm	64 27 .	m 18 . 55 D
KNW	56 22 .	W 12 . 35 D top Pole 7 ft.
KNL	49 7 .	L 10 . 55 D top Pole 6 ft. 8 in.
KNY	18 50 .	Y 9 . 37½ D top
KNγ′	2 15 .	γ′ 7 . 30 D
KNE	1 15½ .	E 6 . 38½ D
KNδ′	9 37½ .	δ′ 6 . 25½ D
KNU	41 4½ .	U 10 . 12½ D top
KNS	60 44 .	S 18 . 41 D
KNR	74 48½ .	R 19 . 20 D
KNB″	104 30 .	B″ 17 . 35½ D
		K 6 . 43½ D top

At K the E. cairn.	Theodolite = 4 ft. 8 in.	
NKO	44 51¾	by reduction
NKR	64 5¾	by reduction
NKP	51 54⅓	N 6 . 41 E
NKn	0 45 .	n 6 . 36 E top
NKF	43 32½ .	F 9 . 46 D top Pole 4 ft. 2 in.
NKG	51 40 .	G 10 . 16 D top Pole 4 ft. 4 in.
		G 10 . 14½ D
NKα	55 12	α 10 3 D
NKH	81 28	H 12 7 D top Pole 6 ft. 5 in.
NKH	81 27½ .	H 12 . 5½ D
NKW	97 18 .	W 11 . 28 D top Pole 7 ft.
NKL	109 19¾ .	L 10 . 25 D top Pole 6 ft. 8 in.
NKL	109 18½ .	L 10 . 23½ D
NKY	153 40 .	Y 10 . 30 D top
NKV	174 20 .	V 9 . 5½ D top
NKE	156 18	E 3 34 D top
NKα′	11 36 .	α′ 6 . 31½ E
NKM′	7 8 .	M′ 4 . 51½ E
NKδ′	6 2⅗ .	δ′ 6 . 53 E
nKα	54 26½ .	α 10 3½ D
nKα	54 25½ .	α 10 . 3½ D
nKβ	39 16½ .	β 8 . 58½ D
α′Kβ′	37 39	α′ 6 32 E

At α the E. end of the N. base.	Theodolite = 4 ft. 4 in.	
γαK	106 0 .	K 10 . 1½ E

	° ′	° ′
$\gamma\alpha$G	101 3 .	G 9 . $37\frac{1}{2}$ E
$\gamma\alpha$N	89 41 .	N 13 . $45\frac{1}{2}$ E
$\gamma\alpha n$	89 25 .	n 13 . $49\frac{1}{2}$ E
$\gamma\alpha$F′	47 38 .	F′ 8 . $6\frac{1}{2}$ E
		β 0 19 D
		γ 0 . 0 top
		Pole 3 ft. 2 in.
At β betw. a and γ theends of the N. base.	Theodolite = 4 ft. 8 in.	
$\alpha\beta k$	174 51	k 0 $41\frac{1}{2}$ E
$\alpha\beta t'$	147 . $37\frac{1}{2}$	t' 1 . $25\frac{1}{2}$ E
$\alpha\beta$F′	71 49 .	F′ 10 . $22\frac{2}{3}$ E
$\alpha\beta$D	108 $1\frac{1}{4}$.	D 9 . $17\frac{1}{2}$ E
$\alpha\beta n$	70 $58\frac{1}{2}$.	n 13 . $5\frac{2}{3}$ E
$\gamma\beta$D	71 $57\frac{3}{4}$	D 9 . $17\frac{7}{10}$ E
$\gamma\beta$F′	108 $9\frac{1}{2}$.	F′ 10 . $22\frac{3}{4}$ E
		α 0 . 6 D
At γ the W. end of the N. base.	Theodolite = 4 ft. $8\frac{1}{2}$ in.	
$\alpha\gamma$F′	50 $6\frac{1}{2}$.	F′ 8 . $23\frac{1}{2}$ E
$\alpha\gamma n$	63 $53\frac{1}{2}$.	n 12 . $27\frac{1}{4}$ E
$\alpha\gamma$D	97 $5\frac{3}{5}$.	D 9 . 41 E
		α 0 . 2 D top
		Pole 3 ft. 2 in.
At n the new W. cairn.	Theodolite = 4 ft. 9 in.	
K$n\alpha$	108 $59\frac{1}{2}$.	K 6 . $40\frac{1}{2}$ D
K$n\beta$	128 $35\frac{3}{4}$.	β 13 . $7\frac{1}{2}$ D
KnF′	128 $55\frac{1}{2}$.	F′ 14 . $10\frac{1}{2}$ D
K$n\gamma$	135 $40\frac{1}{2}$.	γ 12 . 29 D
KnD	173 55	D 10 . 45 D
KnN	38 $28\frac{1}{2}$.	N 1 42 D
KnB″	102 $54\frac{1}{4}$.	B″ 17 31 D
DnA	53 49 .	A 12 . $21\frac{1}{2}$ D
DnG	75 24 .	G 17 . $3\frac{1}{2}$ D
DnP″	78 $44\frac{1}{4}$	P″ 20 . $2\frac{1}{2}$ D
		P″ is a pole in a line with P and K.
DnH	104 35	H 14 . $42\frac{1}{2}$ D
DnL	124 34	L 10 56 D
DnK	173 55	α 13 . $51\frac{3}{8}$ D
NnG	132 59	α 13 . 54 D
		Nn was = $93\frac{1}{2}$ feet by the tape measure.

At E the new E. cairn	Theodolite = 4 ft. 9 in.	
	° ′	° ′
NEA	10 $5\frac{1}{2}$.	A . 6 . $22\frac{1}{2}$ E
NEM′	4 $58\frac{1}{2}$.	M′ 4 . $45\frac{1}{2}$ E
NED	4 $51\frac{1}{2}$.	D 6 . $41\frac{1}{2}$ E
NEK	22 $14\frac{1}{2}$.	K 2 . $11\frac{1}{2}$ E
NEH	78 $40\frac{1}{2}$.	H 11 . $47\frac{1}{2}$ D top
		Pole 6 ft. 5 in.
NEW	94 17 .	W 11 . $16\frac{1}{2}$ D top
		Pole 7 ft.
NEL	106 $21\frac{1}{2}$.	L 10 . 17 D top
		Pole 6 ft. 8 in.
NEY	151 16 .	Y 10 . 37 D
NEV	172 20 .	V 9 . 15 D
NET	172 $22\frac{1}{2}$.	T 9 . $41\frac{1}{2}$ D top
NEZ	144 46 .	Z 9 . $18\frac{1}{2}$ D
NEU	114 $20\frac{1}{2}$.	U 10 . $4\frac{1}{2}$ D
a'Ea'	68 $35\frac{1}{2}$.	a' 6 . 21 E
a'Eδ'	73 $50\frac{1}{2}$.	δ' 6 . $37\frac{1}{2}$ E
a'EN	78 $39\frac{1}{2}$.	N 6 . 33 E
a'Eb'	97 17 .	b' 9 . $5\frac{1}{2}$ D
		a' 8 . 27 D
At M′ the meridian mark on the top of the hill S of P.	Theodolite = 4 ft. $10\frac{1}{2}$ in.	
KM′E	3 $28\frac{1}{2}$.	E 4 . $56\frac{1}{2}$ D
KM′γ'	6 $37\frac{1}{2}$.	γ' 6 . $3\frac{1}{2}$ D top
KM′L	53 $27\frac{1}{2}$.	L 10 . 19 D top
		Pole 6 ft. 8 in.
KM′W	63 $12\frac{1}{2}$.	W 11 . 41 D top
		Pole 7 ft.
KM′H	78 $4\frac{1}{2}$.	H 12 . $56\frac{1}{2}$ D top
		Pole 6 ft. 5 in.
KM′m	86 $33\frac{1}{2}$.	m 22 . 12 D
KM′p	87 54 .	p 22 . 10 D
KM′G	108 49 .	G 12 . 33 D top
		Pole 4 ft. 4 in.
KM′F	118 $3\frac{1}{2}$.	F 12 . 25 D top
		Pole 4 ft. 2 in.
KM′δ'	176 $1\frac{1}{2}$.	δ' 12 . 8 E
		K 5 5 D
At t the centre of the transit instrument near the N. obser.	Theodolite = 4 ft. $8\frac{1}{2}$ in.	
M″tP′	32 24 .	P′ 9 . $18\frac{1}{2}$ D
M″tG	32 $57\frac{1}{2}$.	G 5 . $58\frac{1}{2}$ D

	° ′	° ′
M″tm	5 14 .	m 12 · 32½ D
M″tH	14 28 .	H 7 . 44½ D
M″tW	35 23½ .	W 5 . 47½ D
M″tL	46 22 .	L 4 . 35½ D
mtp	68 13 .	p 12 . 50½ E
mtM′	68 15 .	M′ 22 . 5 E
mtG	144 17 .	G 5 . 57½ D
mtH	96 52 .	H 7 . 43½ D
mtW	75 56 .	W 5 . 47½ D
mtL	64 58 .	L 4 . 37½ D
mtP	54 4½ .	P 17 . 42½ D top
		m 4 . 3 E

M″ bears 1′ 8″ W. of North.
M′ bears South.
P′ in a line with K and P.
p, a pole immediately above or South of the transit instrument.

At p	Theodolite = 4 ft. 5 in.	
GpM′	147 5 .	M′ 22 . 8½ E
Gpδ′	133 6½ .	δ′ 24 . 18 E
GpF	26 43	F 5 26 D
GpP′	0 34½ .	P′ 9 . 23½ D
GpM″	32 47¼ .	M″ 13 . 25½ D
Gpt	32 48	t 19 17 D
Gpm	38 0½ .	m 12 . 34½ D
GpH	47 11 .	H 7 . 49½ D
GpW	68 3½ .	W 5 . 51⅓ D
GpP	76 55½ .	P 13 . 49½ D
GpL	79 0½ .	L 4 . 39½ D
Gpm′	121 24 .	m′ 4 . 50¼ D
GpE	179 37 .	E 17 . 39½ E

m′ is a pole a little above or S. of P.

At m′		
Gm′M′	145 4½ .	M′ 22 . 8½ E
Gm′δ′	131 11 .	δ′ 24 . 12½ E
Gm′p	57 53½ .	p 2 47 D
Gm′t	34 43	t 10 15 D
Gm′F	13 1 .	F 5 . 22½ D
Gm′P′	0 29½ .	P′ 19 . 19 D
Gm′M″	32 29½ .	M″ 13 . 23¼ D
Gm′P	37 6 .	P 18 . 16½ D
Gm′m	37 49½ .	m 12 . 36½ D
Gm′H	47 10 .	H 7 . 49 D
Gm′W	68 11 .	W 5 . 49½ D
Gm′L	79 17½ .	L 4 . 40½ D
Gm′E	177 53 .	E 17 . 54 E

At m		
PmK	15 57	K 15 0 E
Pma	37 11	P 12 27 E

At α′ the S.-west pole.	Theodolite = 4 ft. 11 in. ° ′	° ′
Kα′N	149 3 .	K 6 . 43½ D
Kα′B′	131 3 .	B′ 15 . 51¼ D
Kα′R	102 7¼ .	R 20 . 9 D
Kα′O	73 39 .	O 33 . 52 D
Kα′β′	28 4¼ .	β′ 15 . 56½ D top
Kα′γ′	4 53⅝ .	γ′ 7 . 35⅛ top
Kα′δ′	65 24 .	δ′ 3 41½ top
		Pole 17 ft. 4 in.
Eα′d′	13 48 .	d′ 12 . 54½ D
Eα′a′	8 36½ .	a′ 8 . 3½ D
Eα′K	2 49½ .	E 6 30 D

At β′ the S.-east pole.	Theodolite = 4 ft. 9 in.	
α′β′O	56 47 .	O 21 . 45½ D
α′β′B′	62 21½ .	B′ 10 . 28½ D
α′β′R	84 46 .	R 15 . 17 D
α′β′M	96 41 .	M 17 . 3½ D
α′β′K	114 8½ .	α′ 15 . 42½ E

At γ′ the N.-east pole.	Theodolite = 4 ft. 4 in.	
Nγ′P	56 55¾ .	P 18 . 3⅝ D
Nγ′t	55 22½	t 17 49½ D
Nγ′n	0 50⅞ .	n 7 15 E
Nγ′δ′	8 17¼ .	δ′ 7 . 59¼ E
Nγ′α′	14 17 .	α′ 7 . 31 E
Nγ′K	151 54½ .	K 1 . 34½ E
		N 7 . 23½ E

At δ′ the N.-west pole.	Theodolite = 4 ft. 4 in.	
Kδ′M′	2 53 .	M′ 12 . 36½ D
Kδ′γ′	4 29 .	γ′ 8 . 8½ D
Kδ′m′	73 20½ .	m′ 24 . 9½ D top
Kδ′p	74 35½ .	p 24 . 16½ D top
Kδ′α	108 48½ .	α 12 . 8½ D top
		Pole 4 ft. 4 in.
Kδ′N	164 19¼ .	N 6 . 8½ D top
Kδ′a	108 56 .	K 7 . 3 D
Kδ′E	2 30¼ .	E 6 . 49 D
Eδ′a′	9 5½ .	a 8 . 50½ D
γ′δ′a′	113 20½ .	
γ′δ′t	70 14½ .	t 24 20½ D

At F′	Theodolite = 4 ft. $7\frac{1}{2}$ in.	
	° ′	° ′
DF′n	56 $24\frac{7}{10}$.	
DF′P	77 $37\frac{1}{2}$.	
DF′α	174 $40\frac{1}{4}$.	α 8 . $12\frac{4}{5}$ D
DF′β	124 47 .	β 10 . $30\frac{1}{5}$ D
DF′γ	103 4 .	γ 8 . $28\frac{3}{5}$ D
DF′t'	77 42	D 7 . 11 E
At t'	Theodolite = 4 ft. 10 in.	
$\beta t'$F′	57 7	β 1 . $37\frac{1}{2}$ D
$\beta t'k$	115 32	k 0 41 D
F′$t'k$	172 38	F′ 7 . $50\frac{1}{2}$ E
At k	Theodolite = 4 ft. $10\frac{1}{2}$ in.	
$t'k$F′	4 42	F 5 11 E
$t'k\alpha$	34 47	α 0 27 D
$t'k\beta$	37 13	β 0 52 D
		t' 0 26 E
At a	Theodolite = 4 ft. $8\frac{1}{2}$ in.	
baK	113 $57\frac{1}{2}$	K 14 45 E
baN	155 24	Top of the cairn.
baw	80 40	
baL	59 48	
At b	Theodolite = 4 ft. 6 in.	
abK	43 35	K 13 32 E
abN	18 49	N 11 26
		a 5 39
At d	Theodolite = 4 ft. 9 in.	
cdN	14 11	N 11 33 E
cdG	34 53	
cdH	63 27	H 6 36 D
cdL	111 12	L 6 42 D

At c	Theodolite = 4 ft. 8 in.	
	° ′	° ′
dcL	49 34	L 7 . $21\frac{1}{2}$ D
dcH	97 9 .	H 9 . $17\frac{1}{2}$ D
dcG	136 $9\frac{1}{2}$.	G 4 . 58 D
dcK	118 $55\frac{1}{2}$.	K 14 . $35\frac{1}{2}$ E
dcF	144 20	F 4 24 D
At a'	Theodolite = 4 ft. 10 in.	
E$a'\alpha'$	102 48 .	α' 7 . 56 E
E$a'\delta'$	97 $5\frac{1}{2}$.	δ' 8 . 10 E
E$a'b'$	65 $58\frac{1}{2}$.	b' 6 . $4\frac{1}{2}$ D
E$a'c'$	88 6 .	c' 11 . $20\frac{1}{2}$ D
E$a'd'$	108 $21\frac{1}{2}$.	d' 2 . $34\frac{1}{2}$ E
		E 7 19 E
At b'	Theodolite = 4 ft. $9\frac{1}{2}$ in.	
$c'b'a'$	36 $41\frac{1}{2}$.	a' 5 . $44\frac{1}{2}$ E
$c'b'$E	53 $26\frac{1}{2}$.	E 8 . $41\frac{1}{2}$ E
		c' 4 41 D
At c'	Theodolite = 4 ft. $8\frac{1}{2}$ in.	
$a'c'b'$	121 $12\frac{1}{2}$.	b' 6 . 1 E
		a' 9 . 6 E
At d'	Theodolite = 4 ft 5 in.	
$a'd'\alpha'$	169 $14\frac{1}{2}$.	α' 12 . $35\frac{1}{2}$ E
		a' 2 48 D

Several other angles and bearing of objects were taken, which, being of no use in computing the attraction of the hill, are here omitted. The foregoing tables, containing all the angles collected together, which were observed at one and the same point, include all the horizontal angles that were at different times taken, for ascertaining the relative places, of the principal points and objects, on a horizontal plane. The numerous other angles used, in finding the sections of the ground, are given hereafter, with their computed results annexed to them. We now proceed to speak of the two principal bases, which were accurately measured, as foundations on which every thing else must depend: and first,

Of the measure of the base RB″ *in Glenmore, the valley to the South of Shichallin, taken the* 16*th, &c, of Sept.* 1774.

Here A and B are the names of the two measuring rods, which were laid down alternately in the order as expressed in the following table of measures. The lengths of these rods, by the brass standard, when the thermometer was at 62¼, were thus, viz,

$$\left.\begin{array}{l}\text{A} = 20 \text{ feet } 1\cdot255 \text{ inch.} = 20\cdot10458 \\ \text{B} = 20 \text{ feet } 1\cdot323 \text{ inch.} = 20\cdot11025\end{array}\right\} \text{feet.}$$

The numbers following each rod, with the sign + interposed, are inches and decimal parts; and they denote the distance beyond the end of each rod to the beginning of the next following rod; and therefore the sum of all these numbers must be added to the sum of the lengths of the rods themselves, for the total of the measures. Also, as the first rod began at 2 feet 8 inches from the point R, this number is to be added to the total last mentioned, to give the measure of the whole base from R to B″.

A + 8·29	B + 5·42	B + 8·93	A + 6·18	B + 3·67	A + 3·16	B + 4·65
B + 2·53	A + 7·42	A + 5·39	B + 4·19	A + 5·12	B + 4·64	A + 4·07
A + 6·11	B + 8·14	B + 5·20	A + 4·51	B + 1·06	A + 3·26	B + 3·23
B + 6·66	A + 8·77	A + 3·54	B + 3·04	A + 5·96	B + 4·18	A + 4·16
A + 2·79	B + 3·45	B + 1·26	A + 4·37	B + 2·47	A + 4·04	B + 5·73
B + 1·20	A + 3·80	A + 3·20	B + 2·96	A + 3·84	B + 2·92	A + 4·12
A + 2·07	B + 6·64	B + 5·34	A + 2·47	B + 5·57	A + 3·10	B + 4·91
B + 4·30	A + 7·76	A + 3·74	B + 3·90	A + 2·63	B + 5·11	A + 3·18
A + 0·00	B + 3·28	B + 7·22	A + 5·78	B + 7·41	A + 4·61	B + 3·91
B + 1·78	A + 4·87	A + 1·91	B + 3·97	A + 3·11	B + 3·34	A + 5·28
A + 3·29	B + 6·18	B + 4·46	A + 4·87	B + 1·74	A + 2·57	B + 2·90
B + 2·85	A + 8·70	A + 1·95	B + 4·83	A + 2·07	B + 5·80	A + 4·39*
A + 6·39	A + 7·87	B + 2·26	A + 3·34	B + 4·33	A + 3·37	B + 4·37
B + 4·86	B + 4·75	A + 4·54	B + 3·65	A + 5·93	B + 2·58	A + 3·29
A + 6·08	A + 6·56	B + 4·48	A + 6·96	B + 6·36	A + 2·24	B + 2·12
B + 8·58	B + 5·24	A + 3·14	B + 3·07	A + 4·47	B + 3·48	A + 2·95
A + 9·07	A + 7·90	B + 3·38	A + 3·55	B + 3·75	A + 2·95	B + 3·30
B + 1·53	B + 6·32	A + 5·00	B + 2·93	A + 4·74	B + 2·88	A + 2·82
A + 2·28	A + 6·92	B + 4·85	A + 5·53	B + 3·06	A + 3·69	B + 3·97
B + 7·47	B + 7·28	A + 6·12	B + 5·33	A + 2·58	B + 4·07	A + 1·37
A + 2·40	A + 6·34	B + 3·44	A + 4·38	B + 4·15	A + 2·75	B + 0·00

The sum of all these is 74A + 73B + 669·28 inches, or 74A + 73B + 55·773 feet, including the 2 feet 8 inches at the beginning of the measurement.

Now 74A is = 1487·73892
73B is = 1468·04825
the odd parts 55·773
the sum 3011·56017 is = the base unreduced.

But a reduction of this must be here made, according to the state of the thermometer, and for the wearing of the brass 5 feet standard (see Phil. Trans. vol. 58, for the year 1768, pa. 313, &c; or my Abridgment, vol. xii. p. 572). Now the difference between 62° and $62\frac{3}{4}$ being $\frac{3}{4}$, therefore $3011{\cdot}56 \times \frac{232}{1800000 \times 12} \times \frac{3}{4} = 3011{\cdot}56 \times \frac{29}{2700000} \times \frac{3}{4} = 0{\cdot}024$ feet, is the small correction on account of the thermometer, and which, being added, makes the number become 3011·584, for the length of the base as reduced to the state of 62° of Fahrenheit's thermometer. But the brass rod had been $\frac{1}{1000}$th of an inch shortened by wearing, and it was originally $\frac{1}{1000}$th of an inch shorter than the Royal Society's brass standard yard, so that it is now $\frac{1}{500}$dth of an inch shorter than that standard in the length of 3 feet, or $\frac{1}{18000}$th part of the whole; therefore subtracting the $\frac{1}{18000}$th part, or ·167 from the above quantity, there remains 3011·417 feet for the corrected measure of this base, or the true length of the line RB″.

The above measures, as far as to that marked * inclusively, together with 10 feet $10\frac{1}{2}$ inches more, reach to a place to which the assistants had before measured with the tape line, and by it found to be 2844·8 feet; while the measure of the same by the rods is found to be 2839·3 feet. The difference is $5\frac{1}{2}$ feet, a small part of which might be owing to the unstable state of the wooden stands used in the first quarter of the base; but the greater part of this difference is more likely to be owing to the uncertain way of measuring with a tape, which, to say nothing of the ground not being quite level, is liable to be stretched more or less in length with different degrees of tension, and to be variously warped in length by moisture.

Of the measurement of the base $\alpha\beta\gamma$ in Rannoch, to the North-west of the hill of Shichallin.

1. One part of this base was measured twice over, in different ways. The part $\alpha\beta$ was carefully measured on the 8th of October 1774 with a chain, and found to be 63 chains and $40\frac{1}{2}$ links, or 63·405 chains in length.

Now on the 24th of the same month the chain itself was measured, by means of the five-foot brass standard, when the thermometer was at $38°\frac{1}{2}$, and the length found to be 65·94542 feet. Hence then $65{\cdot}94542 \times 63{\cdot}405 = 4181{\cdot}269$, is the length of all the chains; to which adding 1·764, the breadth of the 63 iron pins, the sum is 4183·033 for the length of $\alpha\beta$ uncorrected.

But $62 - 38\frac{1}{2} = 23\frac{1}{2}$; therefore $-23\frac{1}{2} \times \frac{20}{1700000} \times 4183 = -1{\cdot}056$, is the reduction on account of the state of the thermometer, which being applied with its proper sign, there results 4181·977; and from this last number deducting again the $\frac{1}{18000}$th part or ·232, on account of the wearing of the brass standard, there then remains 4181·745 feet, for the length of the part $\alpha\beta$ of the base in Rannoch, as measured by the chain.

But as the chain was measured not at the same time with the base, but between two and three weeks later, when the air was probably cooler, the reduction above made for the state of the thermometer is perhaps something too great, and we may safely conclude $\alpha\beta$ to be equal 4182 feet as measured by the chain.

2. The whole base $\alpha\beta\gamma$ was next, on the 10th, 11th, and 12th of October, very carefully measured by the twenty-foot measuring rods. The rods at that time measured thus,

$$\left.\begin{array}{l} \text{A} = 20 \text{ feet} + 1{\cdot}306 \text{ inch.} = 20{\cdot}108\frac{5}{6} \\ \text{B} = 20 \text{ feet} + 1{\cdot}354 \text{ inch.} = 20{\cdot}112\frac{5}{6} \end{array}\right\} \text{feet};$$

the thermometer being then at 40°. The number of rods and the additional parts were as follow.

A+4·49	A+3·65	B+2·17	B+1·27	A+1·97	B+1·55	A+2·99
B+3·29	B+3·51	A+2·13	A 2·38	B 1·77	A 1·97	B 2·06
A+6·57	A+3·88	B+2·17	B 2·36	A 2,63	B 2·04	A 2·36
B+3·62	B+2·29	A+3·12	A 2·57	B 3·18	A 2·56	B 1·77
A+3·84	A+3·13	B+2·70	B 2·07	A 2·09	B 2·15	A 1·66
B+3·52	B+3·71	A+3·17	A 2·48	B 1·74	A 2·26	B 1·86
A+4·50	A+3·13	B+3·22	B 2·31	A 2·74	B 2·37	A 2·35
B+3·62.	B+3·13	A+3·16	A 2·28	B 4·49	A 1·94	B 2·65
A+4·88	A+5·43	B+2·07	B 3·96	A 2·60	B 1·97	A 2·14
B+2·74	B+3·08	A+2·22	A 4·87	B 2·45	A 1·94	B 2·35
A+3·24	A+3·57	B+4·83	B 2·61	A 1·41	B 2·77	A 2·43
B+4·30	B+5·68	A+2·39	A 2·22	B 2·67	A 2·09	B 2·59
A+3·50	A+3·89	B+2·68	B 1·63	A 1·98	B 2·14	A 2·48
B+3·26	B+2·78	A+2·09	A 1·87	B 2·96	A 2·45	B 2·48
A+2·96	A+3·27	B+1·97	B 2·41	A 2·27	B 2·98	A 2·91
B+3·32	B+1·84	A+1·48	A 2·62	B 2·60	A 2·40	B 2·47
A+5·93	A+0·00	B+3·20	B 2·09	A 3·08	B 2·50	A 1·80
B+5·00	A+3·14	A+2·62	A 2·27	B 2·23	A 2·82	B 1·99
A+3·43	B+1·73	B+2·37	B 3·02	A 4·70	B 2·37	A 2·81
B+3·87	A+2·41	A+2·47	A 2·41	B 2·28	A 2·76	B 2·44
A+6·37	B+2·99	B+3·48	B 2·53	A 1·79	B 2·91	A 2·07
B+3·48	A+2·62	A+3·16	A 1·84	B 1·83	A 2·58	B 2·44
A+4·86	B+2·18	B+3·50	B 2·68	A 2·74	B 2·34	A 2·11
B+4·87	A+2·72	A+2·33	A 2·37	B 2·48	A 2·36	B 2·49
A+3·08	B+3·02	B+2·37	B 1·87	A 2·23	B 2·67	A 2·11
B+3·67	A+2·46	A+2·68	A 2·38	B 1·66	A 2·19	B 3·08
A+3·28	B+3·68	B+2·68	B 2·67	A 3·08	B 2·37	A 2·67
B+3·43	A+2·62	A+2·70	A 2·07	B 2·31	A 2·93	B 2·43
A+4·82	B+2·72	B+2·68	B 1·84	A 2·20	B 2·37	A 1·93
B+4·46	A+3·33	A+2·05	A 4·03	B 3·06	A 2·86	B 2·75
A+3·99	B+2·93	B+3·09	B 1·72	A 2·36	B 2·42	A 1·99
B+3·13	A+3·14	A+2·50	A 2·14	B 2·24	A 2·16	B 2·17
A+3·55	B+2·93	B+2·52	B 1·80	A 1·66	B 2·21	A 1·83
B+4·19	A+2·40	A+2·62	A 1·27	B 1·53	A 2·75	B 1·93
A+4·06	B+2·06	B+2·43	B 1·88	A 1·80	B 2·03	A 1·65
B+3·94	A+3·13	A+3·01	A 2·36	B 3·53	A 2·73	B 1·04
A+3·64	B+2·57	B+2·43	B 2.04	A 2·45	B 2·14	A 1·96
B+3·23	A+2·85	A+1·97	A 2·37	B 0·00*	A 1·57	B 2·77
A+3·76	B+2·87	B+1·74	B 1·77	A 2·36	B 1·77	A 2·25
B+2·56	A+4·12	A+2·84	A 1·66	B 2·75	A 2·21	B 1·79
A+3·38	B+2·20	B+3·65	B 2·26	A 1·77	B 2·11	A 0·00
B+3·29	A+2·68	A+1·62				

Of the foregoing measures, the sum of all, from the beginning to that marked * inclusively, together with 13 feet 2 inches more, brings us to the point β, before measured to by the chain. Now to this place, by adding together the measures, there are found to be 103A and 102B, and the sum of the parts is 586·71 inches.

$$
\begin{array}{lrl}
\text{Then } 103\text{A} = 103 \times 20{\cdot}108\tfrac{1}{6} & = & 2071{\cdot}210 \\
102\text{B} = 102 \times 20{\cdot}112\tfrac{1}{6} & = & 2051{\cdot}509 \\
586{\cdot}71 \text{ inches} & = & 48{\cdot}893 \\
13 \text{ ft. } 2 \text{ inch.} & = & 13{\cdot}167 \\
\hline
\text{Hence } \alpha\beta\text{, unreduced, is} & & 4184{\cdot}779
\end{array}
$$

But since $62-40 = 22$, therefore the reduction for the state of the air is $-22 \times \frac{29}{1700000} \times 4185 = -\cdot989$, which being applied to the above sum, there remains 4183·79 as corresponding to the state of 62° of the thermometer. From this last number deduct its $\frac{1}{18000}$th part, viz, ·232, and there results 4183·558 for the correct length of the part $\alpha\beta$, as determined by this very accurate method; which is but about a foot and a half more than what it was found to be by the less accurate measure by the chain, which is a nearer approach to an equality than could well be expected.

To determine now the whole length of the base $\alpha\gamma$; by taking the whole sums there are found to be 146A with 144B and 779·78 inches of the odd parts.

$$\begin{array}{lrr} \text{Now } 146\text{A} = 146 \times 20{\cdot}108\tfrac{5}{6} & = & 2935{\cdot}890 \\ 144\text{B} = 144 \times 20{\cdot}112\tfrac{5}{6} & = & 2896{\cdot}248 \\ 779{\cdot}78 \text{ inches} & = & 64{\cdot}982 \\ \hline \text{The sum or } \alpha\gamma\text{, unreduced, is} & & 5897{\cdot}120 \end{array}$$

The correction for the thermometer is $-22 \times \frac{29}{1700000} \times 5897 = -1\cdot394$, which being applied to the number above, there results 5895·726; and this again being diminished by its $\frac{1}{18000}$th part, or ·327, there remains 5895·399 feet, for the correct measure of the base $\alpha\gamma$ in the vale of Rannoch.

There is no occasion here to explain the manner of measuring these two bases by the twenty-foot rods, as that has been very circumstantially done in vol. 65 of the Phil. Trans. for the year 1775.

The following shorter lines were also measured, as they happened to be wanted in different parts of the survey.

		Ft.	In.	
$\alpha'\delta'$	=	269	4	nearly horizontal.
Nn	=	93	6	
Ke	=	94	10	
KE	=	240	10	
ac	=	9	9	
an	=	7	10	
cn	=	1	11	

$$\left.\begin{array}{rrr} & \text{Ft.} & \text{In.} \\ ma = & 70 & 11 \\ mt = & 68 & 3 \\ mp = & 63 & 4 \\ pt = & 27 & 2 \end{array}\right\} \text{not horizontal.}$$

The other measures that were taken, for determining the sections, will be delivered afterwards, when the results or computed altitudes have been obtained, in order to be placed opposite their correspondent angles.

Having now obtained, to a great degree of accuracy, the measured lengths of two lines, which were to serve as bases for all the future calculations; the next consideration was how to make the properest use of them. Every other line or distance, drawn or conceived to be drawn, must be calculated from them, by the help of the angles observed, either at their extremities, or at all the other points and stations in the survey and plan. As these two bases are situated in the low parts of the country, from which but a very few of the other principal stations are visible, one method evidently is, to compute immediately from these bases, such of the great lines in the survey whose extremities are visible from them; and then, from these calculated lines, to compute others next to them, and so on quite around and within the whole figure. In this manner, several values of each line will arise, both from the double computations by the two measured bases, and from the various sets of triangles, which can be formed from the very numerous horizontal angles, which were observed at the several stations. But in this mode of computation, after great labour and pains, I had frequently the mortification to find, that the several values of the same lines would differ so greatly from one another, that it was often very doubtful whether I could rely on any of them, or even on the mean among them all. These differences arose from the small errors in the observed angles, which in some degree are unavoidable; and indeed they were so small, that the sum of the angles of the several triangles, which were used in the calculation, seldom differed by more than a minute or two from 180°. But in a

long connected chain of triangles, dependent on one another, the effects of such small errors at length become too great to be tolerated, in a computation requiring much accuracy.—Another method is, first to compute, from both bases, the length of the line KN, extended along the ridge of the hill from east to west, and from it, as a secondary base, to compute all the other lines in the plan. This method admits of much more accuracy than the former, supposing this secondary base to be truly assigned; because that, from the elevated and central situation of this line, all or most of the other points in the survey are visible, from one or both of its extremities; by which it happens, that the other lines are mostly determinable from it alone, without so close a connection with one another as in the other method of computation. By both of these methods then, and by all the triangles furnished by each of them, I computed all the principal lines in the plan, and either took a mean among the several values of each, or else selected out of them such as from various circumstances I judged it safest to rely on, as nearest the truth. The trigonometrical computations were always accurately made, by the common numbers, and generally repeated by logarithms, and the result of every proportion determined to two or three places of decimals. I shall here abstract the mean or corrected values of some of the principal lines, or horizontal distances, so computed, as well as the secondary base KN, from the Eastern to the Western cairn.

The mean among a great number of ways of computation from the South base, gives the horizontal distance of the secondary base, from K to N, = 4052·2, and the mean of all the results from the North base $\alpha\beta\gamma$ gives KN = 4058·9, and the mean between these two gives 4055·5 for the mean distance between K and N. And this value of KN was used in computing most of the other lines, whose mean results are as here follows.

$\alpha\gamma$ = 5895·4 the Northern base in Rannoch.
RB″ = 3011·4 the Southern base in Glenmore.
NK = 4055·5 the distance of the two cairns.

RA = 5670	NR = 5545	KR = 5952	OR = 3582
AB = 1489	NB″ = 6053	KF = 8227	OB″ = 5466
BC = 4506	NA = 5941	KG = 8036	OA = 6769
CD = 775	NB = 6573	KH = 7748	OS = 3271
DF = 7388	NC = 7797	KW = 7603	OX = 4079
FG = 1166	ND = 7657	KL = 8335	OU = 6061
GH = 4068	NF = 5980	KY = 10008	OZ = 9073
HW = 2118	NG = 6370	KV = 10215	OM = 3317
WL = 1816	NH = 8195	KO = 2615	
LY = 7085	NW = 9059	KP = 3221	
YV = 3636	NL = 10405	Kα = 13710	MS = 381
VT = 2645	NY = 13752	Kβ = 15404	Te = 1335
TZ = 4393	NS = 5795	KM′ = 1817	ze = 3719
ZU = 4132	NO = 2875	Kδ' = 2528	F′D = 6430
UX = 1984	NP = 3271	Ka = 3326	F′F = 3934
XS = 2378	Nα = 11876	Kb = 4409	F′t' = 4098
SR = 1410	Na = 5899		$t'k'$ = 2327
	Nb = 7614		M′h = 1172
	Nh = 3381	PG = 4815	ab = 1843
	Nδ' = 1585	PH = 5196	cd = 1750

From the first three lines, or bases, and the horizontal angles observed at the several stations, a very large and accurate plan of the whole survey was constructed, forming a map of 4 feet long by 4 feet broad, which was verified in every part by the measures of the computed lines, both those above-mentioned, and others, and they were generally found to agree very exactly, according to the scale by which the plan was constructed. The use of this large map was, to receive and admit of the distinct and accurate exhibition of the figures in their true places, expressing the number of feet in elevation or depression, with respect to each observatory, of every point and section of the ground, whose elevation or depression might be observed. But before proceeding to the computation and construction of the points in the sections, we may here abstract the numbers which express the relative elevation of the principal original points in the survey, being the extremes of the lines whose lengths are above abstracted.

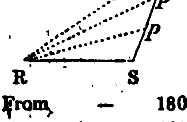
These few numbers are the results of the calculation of several hundreds of triangles, conceived in a vertical position, their bases being either the horizontal lines above-mentioned, or other lines drawn as diagonals between many distant points in the survey, according to the number of vertical angles which had been observed; and of these bases, whether real or imaginary, each generally afforded two vertical triangles, as the angles of elevation and depression were taken alternately at both ends of the lines. It is scarcely necessary to remark, that all these triangles are right-angled, the common base being one of the sides about the right angle, and the other the difference in altitude between the two given points or extremes of the base; and this difference in altitude is found from the application of this proportion, as radius is to the tangent of the angle of elevation or depression, so is the given base, to the altitudinal difference between the two given points, exclusive of the height of the theodolite or other instrument, which was afterwards allowed for. From the resolution of all these triangles, and taking the means of the many corresponding results, were obtained the following numbers, which show how many feet the points denoted by the letters standing against them, are below the level of the point N, or the Western cairn, on the ridge of the hill. They are all referred to this point N, at the Western extremity of the ridge of the hill, because it is the most elevated point in the whole survey.

O	1184	γ	2898	H	2143	U	1613	e	2145
P	1457	A	1303	W	2024	X	1996	M	1958
K	480	B	1313	L	2006	S	1964	M′	322
R	1948	C	1384	Y	2335	a	1012	F′	2246
B″	1920	D	1445	V	2119	b	823	t'	2815
α	2898	F	1904	T	2114	c	1364	k	2835
β	2901	G	1935	Z	1815	d	1539	δ'	172

These depressions, and those of several other principal points, were first carefully computed by means of various different bases, as so many places from which the sections were to commence.

These sections are very numerous, and were made in all directions from the primitive points before-mentioned, and many of them extended to great distances, indeed far beyond the bounds of the plan here annexed, so as to include the nearest hills and valleys of the surrounding country. They were mostly made in vertical planes, in the manner described in the article of the Phil. Trans. before referred to, excepting some few, which are level sections, in planes parallel to the horizon, and some indeed irregular, being neither vertical nor horizontal. To compute the relative altitude of each point in these sections, it is evident, requires the resolution of two different triangles, viz. a horizontal triangle, by which its place in the plan is ascertained, and a vertical triangle, of which one side is the elevation or depression of the point. Of these sections, there are above 70, containing near 1000 points, whose places in the plan and relative altitudes have been computed: so that the number of triangles, whose numeral resolutions have been performed in the course of this business, amounts to several thousands.

Before the abstract of the computation of the sections, we may here set down at large the calculation of one of them, to show the manner in which they have been computed, in the readiest and easiest way which occurred, preserving at the same time the proper degree of accuracy. I shall for this purpose select the third section, as not containing so many poles as some of the others. This section commences at S, and is carried up the hill in a vertical plane, making an angle of 105° with the line RS. The direction of this plane is here represented by the line *sppp*, making with RS the angle RS*p* = 105°. The points *ppp* &c, mark the places of the poles, whose angles of elevation or depression were taken at S, with a proper instrument, and they are written in the second column of the table in this example. At R were observed the several horizontal angles, which lines supposed



From	—	180°
Take	$\angle S =$	105
Leaves	$\angle R + \angle p =$	75

to be drawn from thence made with RS, and these are placed in the third column. And since, in every triangle RSp, the angle S is constant, and the sum of R and p is equal to the constant quantity 75°; therefore each of the angles R, or the numbers in the third column, being subtracted from 75°, there remains the corresponding angle p; and these remainders are placed in the fourth column. Then, since the method of solution is this, as sin. p : sin. R : : RS : Sp $= \frac{\text{sin. R}}{\text{sin. } p} \times$ RS; and again, as radius 1 : tang. elev. : : Sp : alt. of p above S, which will be $=$ Sp $\times$ tang. elev. $= \frac{\text{sin. R}}{\text{sin. } p} \times$ RS $\times$ tang. elev. Or in logarithms sin. R $-$ sin. p $+$ RS $+$ tang. elev. $=$ log. of the altitude of the point. Therefore, having taken, from a table, the sines of R and p, and placed them in the fifth and sixth columns, subtract the latter from the former, and write the remainders in the next or seventh column; to these add the constant logarithm of RS, and write the sums in the eighth column; take out then the tangents of the angles in the second column, and having placed them in the ninth column, add together the adjacent numbers of the eighth and ninth columns, placing the sums in the tenth column, which being the logarithms of the altitudes or depressions of the points p, take the corresponding numbers from a table of logarithms, and set them in the eleventh or last column, for those altitudes or depressions with respect to the point S, with the height of the theodolite included, and which is afterwards allowed for, its height being generally about $4\frac{1}{4}$ or $4\frac{1}{2}$ feet. In the second column, D denotes depression, and E elevation; in the last column, D denotes depression and A altitude.

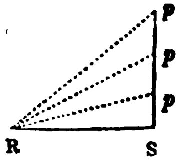
N° of Poles.	Angles of Dep. and El. at S.	Horiz. Angle at R.	75° — ∠'s R, or angles p.	Sines of R.	Sines of p.	Sin. R — sin. p.	Sin. R — sin. p + RS, or sp.	Tang. of Dep. and Elev.	Sum of 8 and 9, or Log. Dep. and Alt.	Dep. & alt. below or above S
1	3° 27′ D	19° 11′	55° 49′	9·51666	9·91763	9·59903	2·74843	8·78022	1·52865	34 D
2	2 36 E	30 24	44 36	9·70418	9·84643	9·85775	3·00715	8·65715	1·66430	46 A
3	4 33	38 28	36 32	9·79383	9·77473	0·01910	3·16850	8·90080	2·06930	117
4	6 12	44 10	30 50	9·84308	9·70973	0·13335	3·28275	9·03597	2·31872	208
5	7 51	47 55	27 5	9·87050	9·61828	0·25222	3·40162	9·13948	2·54110	348
6	10 38	51 22	23 38	9·89574	9·60302	0·29272	3·44212	9·27357	2·71569	520
7	12 20	53 9	21 51	9·90320	9·57075	0·33245	3·48185	9·33974	2·82159	663
8	13 46	54 43	20 17	9·91185	9·53991	0·37194	3·52134	9·38918	2·91052	814
9	15 43	56 21	18 39	9·92035	9·50486	0·41549	3·56489	9·44933	3·01422	1033
10	17 35	57 47	17 13	9·92739	9·47127	0·45612	3·60552	9·50092	3·10644	1278
11	18 0	58 58	16 2	9·93291	9·44129	0·49169	3·64109	9·51178	3·15287	1492
1	2	3	4	5	6	7	8	9	10	11

Thus then every line in the table contains the solutions of the two triangles, the one horizontal and the other vertical, used in finding the altitude of each point or pole in the section. The addition of the constant logarithm of the base RS, to the logarithms in the seventh column, is most easily performed by writing it on the bottom of a small slip of paper, and so sliding it down successively over each of those numbers, then in that position adding them together, and placing the sums immediately opposite, in the next column.

And in this manner were computed the relative altitudes of the points in the other vertical sections; excepting two or

three cases, in which the constant angle formed by the section and the base was a right angle; and one case in which the vertical angles were not taken at the beginning of the section line, but at the other end of the base line, where the horizontal angles were also observed. It may be necessary therefore, to insert and explain an example of each of these cases, and the more so, as they point out the fittest means of measuring these sections, so as to save most part of the labour in the computation, in which the trouble chiefly consists.

Of the case of the right angle, the first section is an instance, where also RS is the base, as before, and the angle RSp being $= 90°$.

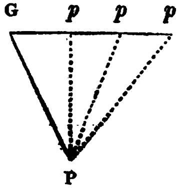


Poles.	Dep. and Elev. at S.	Horiz. Ang. at R.	Tang. of vert. ∠'s at S.	Tang. of ∠'s R.	Sum of Columns 4 and 5.	6th + RS = Log. Alt.	Depth and Alt.
1	5° 16½′ D	10° 0′	8·96533	9·24632	8·21165	1·36105	23 D
2	0 30 E	31 35	7·94086	9·78874	7·72960	0·87899	7½ A
3	4 15	41 56	8·87106	9·95342	8·82448	1·97388	94
4	6 14¼	49 25	9·03861	10·06722	9·10583	2·25523	180
5	8 16	55 51½	9·16224	10·16870	9·33094	2·48033	302
6	10 13	59 57½	9·25582	10·23783	9·49365	2·64305	440
7	11 37	62 56½	9·31297	10·29174	9·60471	2·75411	568
8	12 25	65 3¼	9·34276	10·33257	9·67533	2·82472	668
9	13 21	66 41½	9·37532	10·36568	9·74100	2·89040	777
10	14 10	67 36¼	9·40212	10·38519	9·78731	2·93671	864
11	15 17	68 42½	9·43657	10·40925	9·84582	2·99522	989
12	17 46	70 58	9·50572	10·46221	9·96793	3·11733	1310
13	19 33	72 48	9·55035	10·50927	10·05962	3·20902	1618
14	20 6	74 30	9·56342	10·55701	10·12043	3·26983	1861
1	2	3	4	5	6	7	8

In this form there are three columns less than in the former, by which it happens, that about one-third of the labour is saved. The method of solution is thus; as radius 1 : tang. R :: RS : Sp = RS × tang. R; and again, as 1 : tang. vertical angle S :: Sp : Sp × tang. S = RS × tang. R × tang. S. Or, in logarithms, log. RS + tang. R + tang. S = log. of the vertical perpendicular: and by this theorem, it is evident, the columns of this table are constructed.

But nearly the same saving, in the great labour of computation, would be made, if the vertical and horizontal angles had both been taken at the end of the base farthest from the beginning of the section. And this method would also be much the easiest in making the survey on the ground, as there would then need only one observer, with an instrument to measure both horizontal and vertical angles; and any person, without an instrument, could direct in a line the person who moves and places the poles, or he may even direct himself after his first pole has been placed, by means of a back object, as is commonly done in land surveying.

Of this kind there happens to have been one section taken, proceeding from G, and making with GP an angle of 85°, P being the Northern obervatory, and where both the bearings and depressions of the points p in the section line were observed.



From 180°
Take $\angle$G = 85
$\angle$P + $\angle p$ = 95

	Log.
PG	3·68262
sin. G	9·99834
Sum	3·68096

this gives a constant number from which the sines of p, in the fifth column, are to be deducted.

Poles.	Vertic. Angles at P.	Horiz. Angle at P.	95°— $\angle$P = $\angle p$	Sines of $\angle$'s at p.	PG + s.G — s.p = Pp	Tang. of Depr.	Sum of Col. 6 & 7. = Pp.	Depth below P
	° ′	° ′	° ′					
1	8 45D	9 15	85 45	9·99880	3·68216	9·18728	2·86944	740
2	8 46	16 25	78 35	9·99132	3·68964	9·18812	2·87776	755
3	9 58	27 16	67 44	9·96634	3·71462	9·24484	2·95946	911
4	8 38	30 52	64 8	9·95415	3·72681	9·18136	2·90817	809
5	7 6	34 30	60 30	9·93970	3·74126	9·09537	2·83663	686
6	5 23	37 55	57 5	9·92400	3·75696	8·97421	2·73117	538
1	2	3	4	5	6	7	8	9

Here, it is evident, is a saving of two of the most laborious columns in the table. This happens because that, in every triangle PGp, there are now constant those two parts which enter the proportion made use of in the calculation, viz, PG

and the angle G. For then it is, as sin. p : sin. G :: PG : Pp, or log. Pp = log. PG + sin. G − sin. p; so that the sum of the logarithms of PG and sine of ∠ G, is a constant number, from which the numbers in the fifth column are to be subtracted, to find those in the sixth column. The rest of the work is the same as in the first example.

As to the irregular sections, the computation of them differs so little, in manner, from that of the usual vertical sections, that an example of it is unnecessary: and as to the few horizontal sections, they need no computation, but only an allowance for the height of the theodolite.

In the following abstract, of the results of the computation of the sections, the first column contains the number of the pole, the second and third the vertical and horizontal angles, and the last the difference of altitude in feet, between the foot of each pole, and the point from which the vertical angles were observed, after making the allowance for the height of the theodolite above the ground. At the end of this abstract, is a plate, n°. 1, of the figures referring to the number of the section, showing the direction in which it was carried, with the degrees and minutes in the angle formed by it and the base line.

SECTION 1.

Pole	Vert. ∠'s at s.	Bearings at R.	Diff. of Alt.
	° '	° '	
1	5 16½ D	10 0	18 D
2	0 30 E	31 35	12 A
3	4 15	41 56	99
4	6 14¼	49 25	185
5	8 16	55 51¼	307
6	10 13	59 57½	444
7	11 37	62 56½	572
8	12 25	65 3¼	673
9	13 21	66 41½	782
10	14 10	67 36¼	869
11	15 17	68 42¼	994
12	17 46	70 58	1315
13	19 33	72 48	1623
14	20 6	74 30	1866

SECTION 2.

Pole	Vert. ∠'s at s.	Bearings at R.	Diff. of Alt.
	° '	° '	
1	3 40 D	20 46	30 D
2	1 26 E	32 42	28 A
3	4 20	42 8	103
4	6 21	49 30	192
5	9 58½	59 2	429
6	11 50½	62 30	591
7	12 52	65 2	721
8	13 22	65 51	780
9	13 36½	66 9	806
10	15 12½	67 36	972
11	16 24	68 45	1118
12	17 45	70 0	1302
13	18 45	71 3	1467
14	19 35	71 57½	1625
15	19 59	72 34	1726

SECTION 3.

Pole	Vert. ∠'s at s.	Bearings at R.	Diff. of Alt.
	° '	° '	
1	3 27 D	19 11	29 D
2	2 36 E	30 24	51 A
3	4 33	38 28	122
4	6 12	44 10	213
5	7 51	47 55	352
6	10 38	51 22	524
7	12 20	53 9	668
8	13 46	54 43	818
9	15 43	56 21	1038
10	17 35	57 47	1283
11	18 0	58 58	1427

SECTION 4.

Pole	Vert. ∠'s at R.	Bearings at s.	Diff. of Alt.
	° '	° '	
1	7 42 D	13 8	39 D
2	1 2 E	31 35	19 A
3	4 4	40 24	80
4	5 36	48 56	137
5	6 55	56 56	215
6	9 10	62 36	337
7	9 23	68 12	415
8	10 17	70 33	495
9	11 18	74 0	623
10	13 2	76 3	785
11	13 37	76 47	848
12	14 30	77 45	946
13	15 3	78 58	1042
14	16 15	80 11	1200
15	17 24	81 13	1362
16	18 32	82 5	1528
17	19 29	82 59	1699
18	20 7	83 30	1812

SECTION 5.

Pole	Vert. ∠'s at s.	Bearings at x.	Diff. of Alt.
	° '	° '	
1	3 46 D	15 0	38 D
2	0 14 E	24 23	9 A
3	2 16	36 14	60
4	3 29	46 5	112
5	4 21	54 24	163
6	5 48	63 55	258
7	7 2	71 30	361
8	9 3	76 30	513
9	11 25	81 18	720
10	12 55	84 16	874
11	13 54	87 22	1017
12	14 51	90 30	1182
13	15 9	93 0	1295

SECTION 6.

Pole	Vert. ∠'s at x.	Bearings at s.	Diff. of Alt.
	° '	° '	
1	9 16 D	11 43	80 D
2	4 46 D	16 51	60 D
3	1 10 E	23 50	28 A
4	4 51	30 2	137
5	6 6	33 14	196
6	7 32	37 25	287
7	9 30	40 24	409
8	11 15	43 12	546
9	12 22	45 18	656

	° ′	° ′	
10	13 35	47 9	782
11	15 19	48 54	955
12	16 29	50 12	1093
13	17 8	51 1	1171
14	17 27	51 48	1248

SECTION 7.

Pole	Vert. ∠'s at X.	Bearings at S.	Diff. of Alt.
	° ′	° ′	
1	7 30 D	9 30	60 D
2	1 49	15 52	24
3	2 5 E	20 49	52 A
4	4 30	24 48	135
5	5 41	28 23	208
6	7 13	30 27	297
7	8 14	32 29	380
8	9 24	33 41	465
9	10 26	35 0	558
10	11 41	36 21	678
11	12 36	37 15	773
12	13 31	38 17	885
13	13 46	39 30	973
14	13 56	40 17	1036

SECTION 8.

Pole	Vert. ∠'s at A.	Bearings at B.	Diff. of Alt.
	° ′	° ′	
1	17 56½ D	37 12	516 D
2	10 12½	41 50	359
3	5 41	44 48	229
4	2 25	47 2	107
5	0 29	49 6	20
6	1 12 E	51 0	74 A
7	3 0	52 35½	198
8	5 0	54 44½	378
9	6 18	55 56½	520
10	7 2	56 47½	620
11	8 52	57 24	821
12	10 10	57 55½	986
13	11 24	58 28	1160
14	12 20½	58 58	1316

N. B. The place of this last pole would seem to be the same as N the Western cairn, as the section was directed through it. But then the last number 1316 is too great; as, from all other measures, the diff. in alt. between A and N is only 1303 feet. This diff. of 13 feet seems to be caused by the last bearing being about 7′ too great, for in other places this angle is only 58° 51′. And indeed many other angles taken at the same time with the above seem to be much wrong, as they greatly differ from corresponding ones taken at other times.

Such differences among corresponding angles I often met with in the measures contained in the books of the survey, and it required much care to detect them, and trouble to reconcile them.

SECTION 9.

Pole	Bearings at A.	Bearings at B.
	° ′	° ′
1	132 21½	36 45½
2	130 27	37 34½
3	127 26	39 8
4	124 25½	40 18
5	119 6½	43 19
6	111 37	48 7
7	103 45½	52 58
8	95 25	58 50
9	85 55	67 54
10	78 2½	75 58½
11	73 14	82 30
12	71 41½	86 21
13	70 38½	89 33
14	69 39	92 23

This is a horizontal or level section through A, and therefore each point is 4½ feet (the height of the theodolite) above that point.

SECTION 10.

Pole	Vert. ∠'s at A.	Bearings at B.	Diff. of Alt.
	° ′	° ′	
1	19 41 D	51 12	531 D
2	13 52	57 22½	426
3	8 7½	62 56	295
4	2 30	68 47	106
5	0 59	73 55½	47
6	0 22½ E	76 52	27 A
7	0 59	79 14½	70
8	1 39	81 7	124
9	2 25	82 42	194
10	3 5	84 12	266
11	3 36	85 41½	338
12	3 48	86 27	373

SECTION 11.

Pole	Vert. ∠'s at D.	Bearings at C.	Diff. of Alt.
	° ′	° ′	
1	12 56 D	48 18	231 D
2	9 50	62 4	334

	° ′	° ′	
3	5 15	67 9	242
4	2 25	69 51½	135
5	0 59 E	71 20	70 A
6	3 0	72 13	220
7	4 31	73 9	363
8	6 7	73 53	533
9	6 59	74 27	652
10	7 53	74 52	777
11	8 49	75 9½	904
12	9 27	75 42	1048
13	9 44	76 12½	1168
14	9 53	76 34	1258
15	10 38	76 55½	1443

SECTION 12.

Pole	Bearings at D.	Bearings at C.
	° ′	° ′
1	68 14½	102 6
2	71 16	98 23
3	73 55	94 56
4	79 0	89 33
5	85 28	82 49
6	91 19	76 48
7	97 5	71 8
8	102 21	66 20
9	108 18	60 36
10	113 33	56 15
11	117 12	53 17
12	119 16	52 4
13	119 53	52 1

Each of these poles is five feet above D, the section being horizontal and taken at that point.

SECTION 13.

Pole	Vert. ∠'s at G.	Bearings at F.	Diff. of Alt.
	° ′	° ′	
1	3 7 D	75 2	92 D
2	0 12	79 46	2
3	4 1 E	91 55	221 A
4	6 8	95 0	385
5	7 49	96 34	530
6	9 50	99 27	789
7	10 52	100 10	915
8	11 52	100 44	1040
9	12 54	101 15	1172
10	13 56	101 49	1327
11	14 49	102 20	1472
12	15 10	103 3	1710
13	15 45	103 27	1837
14	16 55	104 20	2013

SECTION 14.

Pole	Vert. ∠'s at G.	Bearings at F.	Diff. of Alt.
	° ′	° ′	
1	2 3 E	97 5	73 A
2	2 35	99 14	95
3	3 52	109 39	182
4	5 58	117 53	384
5	6 25	119 8	440
6	8 31	120 34	633
7	10 14	121 51	827
8	11 29	122 41	985
9	12 30	123 34	1149
10	13 4	124 15	1272
11	13 14	124 41	1339

SECTION 15.

Pole	Vert. ∠'s at G.	Bearings at H.	Diff. of Alt.
	° ′	° ′	
1	9 46 D	14 14	172 D
2	3 32	23 9	102
3	0 45	31 7	27
4	1 43 E	36 15	94 A
5	6 27	45 0	465
6	8 10	47 32	642
7	9 10	49 47	781
8	10 39	51 37	970
9	12 0	53 36	1177
10	13 24	55 38	1422
11	14 0	57 18	1584
12	14 18	58 37	1704
13	14 21	58 58	1734

SECTION 16.

Pole	Vert. ∠'s at H.	Bearings at G.	Diff. of Alt.
	° ′	° ′	
1	7 22 D	21 37	189 D
2	6 11	34 32	248
3	4 1	43 23	201
4	0 9	48 20	4
5	1 43 E	54 38	119 A
6	3 35	60 8	275
7	5 14	65 8	446
8	6 49	70 18	653
9	8 1	73 19	898
10	8 41	76 39	977
11	9 18	78 59	1104
12	10 0	80 22	1246

13	10° 44′	82° 0′	1403
14	11 31	83 24	1572
15	13 8	84 43	1874

There seems to be some general error in this section, as the depressions and altitudes are utterly incompatible with those of all the other neighbouring points in the plan.

SECTION 17.

Pole	Vert. ∠'s at H.	Bearings at G.	Diff. of Alt.
1	12° 15′ D	16° 13′	242 D
2	7 11	25 6	216
3	4 19	33 3	171
4	1 49	38 20	82
5	1 59 E	45 24	119 A
6	3 38	50 3	243
7	5 10	53 33	377
8	7 10	57 3	572
9	8 17	60 46	731
10	8 45	62 58	820
11	9 53	66 56	1038
12	10 24	69 0	1161
13	10 45	70 35	1260
14	11 31	72 15	1424
15	12 10	73 55	1590
16	12 31	75 7	1703

SECTION 18.

Pole	Vert. ∠'s at H.	Bearings at W.	Diff. of Alt.
1	13° 35′ D	21° 40′	186 D
2	9 17½	43 28	263
3	5 21	55 6	203
4	2 4	63 8	95
5	1 23 E	69 50	84 A
6	3 37	73 24	238
7	5 25	76 0	386
8	6 55	78 7	531
9	8 33	81 13	737
10	9 28	88 4	1108
11	9 38	89 10	1201
12	9 53	89 58	1290
13	10 24	90 33	1407
14	11 2	91 10	1552

SECTION 19.

Pole	Vert. ∠'s at W.	Bearings at L.	Diff. of Alt.
1	18° 8′ D	14° 51′	168 D
2	16 28	31 38	388
3	11 53	39 42	394
4	7 47	42 48	292
5	4 11	46 12	180
6	2 10	48 6	100
7	0 31	50 16	23
8	2 24 E	52 45	148
9	5 55	54 48	402
10	6 48	56 30	657

SECTION 20.

Pole	Vert. ∠'s at W.	Bearings at L.	Diff. of Alt.
1	22° 45′ D	7° 37′	98 D
2	19 0	16 20	179
3	18 41	31 8	367
4	14 20	41 54	412
5	9 41	48 4	341
6	3 38	54 6	155
7	1 15	56 50	57
8	0 55 E	60 7	55 A
9	4 4	62 58	257
10	6 45	66 2	487

SECTION 21.

Pole	Vert. ∠'s at L.	Bearings at W.	Diff. of Alt.
1	23° 59′ D	21° 18′	295 D
2	19 25	33 12	377
3	14 37	48 26	448
4	10 5	56 7	383
5	5 9	62 51	237
6	1 2	69 24	56
7	2 2 E	71 36	133 A
8	4 22	73 2	297
9	6 6	75 22	455

SECTION 22.

Pole	Vert. ∠'s at a.	Bearings at A.	Diff. of Alt.
1	7° 16′ D	5° 2′	20 D
2	2 49 E	10 20	25 A

	° ′	° ′	
3	3 33	15 13	44
4	4 18	29 34	120
5	7 0	32 15	222
6	9 1	35 3	331
7	10 26	38 52	471
8	12 19	41 46	657
9	14 40	43 34	874
10	14 45 top of E. cairn.	43 35	874

SECTION 23.

Pole	Vert. ∠'s at *a*.	Bearings at *b*.	Diff. of Alt.
	° ′	° ′	
1	15 2 D	4 33	34 D
2	0 51	15 15	4
3	0 10	35 37	0
4	3 7 E	41 53	97 A
5	6 43	42 34	209
6	7 36	43 30	244
7	9 50	45 48	342
8	11 35	59 5	664

SECTION 24.

Pole	Vert. ∠'s at *b*.	Bearings at *a*.	Diff. of Alt.
	° ′	° ′	
1	3 27 D	10 29	29 D
2	0 32 E	21 9	12 A
3	2 58	32 57	64
4	4 40	58 53	133
5	5 14	76 22	173
6	6 24	121 5	340
7	7 50	125 12	448
8	8 27	130 23	546
9	8 55	134 57	664
10	9 6	136 15	712

SECTION 25.

Pole	Vert. ∠'s at *b*.	Bearings at *a*.	Diff. of Alt.
	° ′	° ′	
1	15 39 D	8 14	74 D
2	1 47 E	52 24	54 A
3	3 9	63 13	115
4	5 51	70 37	248
5	9 11	81 35	505
6	11 21	85 10	691
7	12 44	87 45	802

SECTION 26.

Pole	Vert. ∠'s at *b*.	Bearings at *a*.	Diff. of Alt.
	° ′	° ′	
1	17 53 D	6 35	64 D
2	0 26	48 5	9
3	2 1 A	60 8	93 A
4	3 4	62 45	151
5	4 40	66 15	256
6	6 20	68 29	374
7	8 10	71 57	550

SECTION 27.

Pole	Vert. ∠'s at *b*.	Bearings at *a*.	Diff. of Alt.
	° ′	° ′	
1	4 30 E	59 25	280 A
2	3 33	57 12	204
3	1 47	54 11	92
4	0 10 D	50 15	3 D
5	1 22	46 54	47
6	2 20	42 15	69
7	4 50	37 30	122
8	18 52	6 12	65

SECTION 28.

Pole	Vert. ∠'s at T.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	10 23 D	16 36	142 D
2	9 26	23 56	195
3	5 23	28 45	136
4	4 14	32 24	120
5	2 42	37 46	96
6	1 34	41 20	62
7	0 35½	45 24	24
8	1 30 E	48 52	89 A
9	3 30	51 4	220
10	4 20	54 22	307
11	4 49	56 11	367
12	5 15	58 41	443
13	6 0	60 4	537
14	6 47	62 4	664
15	7 24	63 33	778
16	8 16	64 46	924
17	8 46	65 44	1030

SECTION 29.

Pole	Vert. ∠'s at T.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	8 49 D	26 9	182 D
2	6 51	30 7	163

	° ′	° ′	
3	4 27	38 45	140
4	2 55	44 0	106
5	1 30	49 34	62
6	0 10	52 56	3
7	0 42 E	55 55	42 A
8	2 12	58 28	132
9	3 3	60 58	195
10	3 36	63 32	247
11	4 4	66 16	303
12	4 8	68 28	331
13	4 27	70 9	377
14	5 0	71 54	450
15	5 7	74 23	506

SECTION 30.

Pole	Vert. ∠'s at T.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	3 26 E	81 0	353 A
2	3 10	79 53	313
3	2 55	77 7	263
4	2 46	75 2	233
5	2 35	72 45	204
6	2 26	70 16	179
7	1 20	66 28	91
8	0 19	62 32	23
9	0 49 D	57 58	37 D
10	2 33	51 54	107
11	4 23	43 37	151
12	6 21	35 48	176
13	8 3	29 13	145
14	10 2	21 54	170

SECTION 31.

Pole	Vert. ∠'s at Y.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	11 18 D	10 40	132 D
2	5 29	15 43	93
3	2 56	19 13	60
4	1 42	22 49	40
5	0 42	28 23	20
6	0 16	31 43	5
7	0 30 E	36 10	28 A
8	1 33	40 41	90
9	2 13	45 0	146
10	2 41	47 18	190
11	2 54	50 47	231
12	3 13	53 12	278
13	3 39	56 0	349
14	4 53	58 51	519
15	5 42	60 38	650
16	6 0	62 8	728
17	6 27	63 23	825
18	6 41	64 21	892
19	6 55	65 50	988

SECTION 32.

Pole	Vert. ∠'s at Y.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	7 9 E	51 46	812 A
2	6 55	50 50	751
3	6 7	49 45	631
4	5 0	46 51	452
5	4 25	45 30	377
6	3 29	43 37	275
7	3 5	40 32	214
8	2 17	38 26	146
9	1 21	35 57	80
10	0 52	33 0	47
11	0 3	29 47	7
12	1 3 D	26 24	33 D
13	2 3	23 40	60
14	3 21	18 30	73
15	7 54	13 4	121

SECTION 33.

Pole	Vert. ∠'s at T.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	13 6 D	4 6	108 D
2	9 15	8 4	132
3	7 32	14 53	187
4	5 10	21 28	167
5	3 55	26 54	148
6	2 36	36 19	119
7	1 30	45 32	78
8	0 40	56 5	37
9	0 28 E	65 38	38 A
10	1 40	71 15	132
11	2 22	77 3	197
12	2 25	87 12	226
13	2 32	93 33	257

SECTION 34.

Pole	Vert. ∠'s at T.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	4 43 E	84 3	471 A
2	4 8	79 45	386
3	3 31	76 0	311
4	2 45	71 30	229
5	2 0	66 22	156
6	1 8	60 49	84
7	0 19	53 20	25
8	0 55 D	46 47	46 D
9	1 29	40 15	69
10	1 59	34 52	83
11	3 41	26 15	127

12	5° 27′	21° 55′	165
13	6 0	17 16	150
14	9 5	9 56	143
15	14 6	4 32	109

SECTION 35.

Pole	Vert. ∠'s at T.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	15 16 D	1 58	86 D
2	9 36	5 3	126
3	8 21	8 5	162
4	7 34	12 54	208
5	5 38	17 50	191
6	4 21	22 48	179
7	3 12	32 56	153
8	2 31	41 54	135
9	1 42	54 12	103
10	1 10	64 26	76
11	0 52	75 48	61
12	0 30	86 54	36
13	0 14	94 17	15

SECTION 36.

Pole	Vert. ∠'s at U.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	17 42 D	2 45	60 D
2	15 55	22 26	465
3	12 39	25 13	404
4	8 51	29 37	338
5	7 23	33 36	318
6	3 51	40 33	205
7	1 22	44 45	80
8	1 3 E	48 26	77 A
9	2 54	50 52	218
10	4 22	52 45	347
11	5 37	54 40	471
12	6 30	55 30	552

SECTION 37.

Pole	Vert. ∠'s at U.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	3 53 E	63 50	345 A
2	3 27	60 54	286
3	1 55	57 45	150
4	0 16	54 3	23
5	1 14 D	50 40	74 D
6	2 53	47 28	166
7	4 45	43 0	247
8	6 25	38 49	300
9	8 40	33 14	348
10	12 6	28 25	420
11	16 36	23 43	419

SECTION 38.

Pole	Vert. ∠'s at U.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	15 9 D	20 37	405 D
2	11 30	31 39	438
3	9 25	36 7	398
4	6 46	40 30	314
5	5 27	46 10	283
6	3 32	52 5	204
7	2 49	55 9	171
8	1 49	61 25	122

SECTION 39.

Pole	Vert. ∠'s at U.	Bearings at Z.	Diff. of Alt.
	° ′	° ′	
1	4 28 D	69 7	312 D
2	5 35	59 8	345
3	6 30	52 3	366
4	8 9	45 25	418
5	10 32	38 30	485
6	12 47	34 3	544
7	13 47	29 27	531

SECTION 40.

Pole	Vert. ∠'s at P.	Bearings at P.	Diff. of Alt.
	° ′	° ′	
1	8 45 D	9 15	735 D
2	8 46	16 25	750
3	9 58	27 16	906
4	8 38	30 52	804
5	7 6	34 30	681
6	5 23	37 55	533

SECTION 41.

Pole	Vert. ∠'s at O.	Bearings at A.	Diff. of Alt.
	° ′	° ′	
1	10 41 D	23 33	510 D
2	11 31	32 36	769
3	8 52	37 28	682
4	5 38	44 24	524
5	4 52	46 39	480
6	4 6	51 8	435
7	2 8	56 26	269
8	1 21	58 30	178
9	0 57	59 53	129
10	0 46	62 21	103
11	0 40	64 35	102
12	0 5	68 41	10
13	(10 E	69 48	36 A
14	(38	71 20	109

SECTION 42.

Pole	Vert. ∠'s at M.	Bearings at S.	Diff. of Alt.
	° ′	° ′	
1	0 17 D	73 48	0
2	3 25 E	80 57	72 A
3	4 52	85 4	126
4	6 52	88 53	232
5	8 33	90 25	332
6	10 2	91 34	439
7	11 44	93 0	610
8	13 7	94 0	787
9	15 10	94 34	1002
10	17 21	95 13	1292
11	18 38	95 37	1506
12	19 36	96 2	1736
13	19 38	96 0	1727

SECTION 43.

Pole	Vert. ∠'s at O.	Bearings at A.	Diff. of Alt.
	° ′	° ′	
1	12 3 D	31 52	759 D
2	10 54	36 42	776
3	8 58	41 8	705
4	7 2	46 9	614
5	5 35	49 36	521
6	3 33	58 6	389
7	2 52	62 38	341
8	2 14	67 20	290
9	2 12	85 34	421

SECTION 44.

Pole	Vert. ∠'s at R.	Bearings at S.	Diff. of Alt.
	° ′	° ′	
1	3 41 D	6 37	32 D
2	1 18	10 10	21
3	0 51 E	11 21	25 A
4	0 52	12 20	30
5	1 4	16 25	70
6	2 3	16 52	143
7	3 6	17 27	243
8	4 38	18 0	413
9	5 14	18 6	478
10	5 41	18 11	530
11	6 28	18 27	647

SECTION 45.

Pole	Vert. ∠'s at A.	Bearings at R.	Diff. of Alt.
	° ′	° ′	
1	2 15 D	30 14	122 D
2	2 0	31 57	116
3	1 22	38 35	100
4	0 8 E	42 19	16 A
5	4 38	49 48	517
6	5 12	50 45	598
7	5 58	52 13	721
8	6 37	54 58	878
9	7 18	56 7	1008
10	7 22	56 32	1037

SECTION 46.

Pole	Vert. ∠'s at D.	Bearings at C.	Diff. of Alt.
	° ′	° ′	
1	8 50 D	96 20	201 D
2	7 24	111 4	260
3	6 36	114 46	272
4	6 9	116 22	275
5	5 24	119 44	298
6	5 4	121 17	315
7	4 16	122 24	292
8	3 27	123 13	255
9	2 23	124 17	197
10	1 17	124 55	113
11	0 44 E	125 50	82 A
12	1 17	126 8	146

SECTION 47.

Pole	Vert. ∠'s at T.	Bearings at V.	Diff. of Alt.
	° ′	° ′	
1	10 42 D	17 45	152 D
2	6 6	28 48	145
3	3 19	38 30	111
4	2 4	43 5	79
5	0 44	47 48	30
6	0 30 E	49 46	30 A
7	2 28	52 45	143
8	3 24	54 21	205
9	4 39	59 0	335
10	4 54	62 16	387
11	5 57	64 50	517
12	6 16	65 38	562
13	6 50	67 39	666
14	7 11	69 40	764

SECTION 48.

Pole	Vert. ∠'s at V.	Bearings at Y.	Diff. of Alt.
	° ′	° ′	
1	9 0 D	17 3	164 D
2	7 0	30 56	236
3	3 22	44 36	170
4	1 16	47 23	66
5	0 5	49 52	0

6	2° 46′ E	58° 6′	214 A
7	4 57	64 34	457
8	5 53	67 2	585
9	6 21	70 8	696
10	6 30	72 34	772
11	6 34	75 13	857
12	7 41	77 32	1097
13	8 39	79 6	1316
14	8 46	79 34	1361
15	8 52	80 32	1435
16	9 6	82 53	1641

SECTION 49.

Pole	Vert. ∠'s at T.	Bearings at *e*.	Diff. of Alt.
1	11° 56′ D	16° 58′	83 D
2	9 12	35 25	120
3	8 14	64 20	190
4	8 12	80 0	250
5	5 55	92 52	234
6	4 14	98 36	193
7	1 58	103 12	101
8	0 34 E	106 12	39 A
9	1 20	108 26	92
10	2 15	109 6	157
11	3 7	110 35	230
12	3 49½	111 48	300

SECTION 50.

Pole	Vert. ∠'s at L.	Bearings at W.	Diff. of Alt.
1	17° 46′ D	10° 52′	226 D
2	17 2	13 33	296
3	14 51	16 2	333
4	10 47	20 13	363
5	9 26	23 0	418
6	8 54	24 37	466
7	8 7	26 16	508
8	7 8	27 23	507
9	6 17	28 27	507
10	4 55	29 28	451
11	4 30	30 12	490
12	3 51	30 55	433
13	3 2	31 15	357
14	2 35	31 29	315

SECTION 51.

Pole	Vert. ∠'s at F.	Bearings at F′.	Diff. of Alt.
1	14° 37′ D	3° 57′	73 D
2	2 56	44 8	140
3	1° 43′	48° 47′	90
4	1 3	55 5	61
5	1 0	60 58	65
6	0 41	67 26	50
7	0 23	74 28	31
8	0 25 E	82 0	52 A
9	1 53	83 46	225
10	3 17	87 0	423
11	3 32	87 28	461

SECTION 52.

Pole	Vert. ∠'s at F′.	Bearings at F.	Diff. of Alt.
1	2° 48′ E	68° 41′	187 A
2	3 50	83 0	282
3	6 45	102 0	630
4	7 6	108 57	728
5	8 4	111 38	863
6	8 37	112 58	943
7	9 19	115 56	1075
8	10 8	118 30	1230
9	10 57	120 40	1393
10	11 30	122 30	1528
11	12 22	124 50	1748
12	13 40	127 12	2075
13	14 3	128 13	2205
14	14 7	128 32	2239

SECTION 53.

Pole	Vert. ∠'s at F′.	Bearings at F.	Diff. of Alt.
1	3° 11′ E	64° 3′	208 A
2	5 9	71 24	366
3	5 58	75 50	448
4	6 14	84 12	522
5	8 51	91 23	821
6	9 0	97 47	924
7	10 7	99 40	1074
8	11 2	102 33	1236
9	11 45	104 44	1375
10	12 6	106 28	1469
11	12 43	108 24	1612
12	12 55	109 18	1673

SECTION 54.

Pole	Vert. ∠'s at *t*.	Bearings at F′	Diff. of Alt.
1	0° 22′ E	14° 7′	12 A
2	1 27	20 5	44
3	5 13	39 2	241
4	5 46	45 18	299

Pole	Vert. ∠'s	Bearings	Diff. of Alt.
	° ′	° ′	
5	6 9	55 40	379
6	6 38	62 0	451
7	7 11	70 8	553
8	7 30	78 56	662
9	7 48	84 30	755
10	7 50	94 8	908
11	7 54	98 26	1006
12	7 53	100 58	1077

SECTION 55.

Pole	Vert. ∠'s at t'.	Bearings at r'.	Diff. of Alt.
	° ′	° ′	
1	2 0 E	11 46	35 A
2	2 3	22 45	62
3	4 3	25 32	130
4	7 32	34 49	324
5	9 15	42 55	497
6	9 52	48 8	604
7	10 30	53 40	736
8	10 33	62 34	917
9	11 7	66 12	1060
10	11 57	69 15	1235
11	11 52	73 19	1370

SECTION 56.

Pole	Vert. ∠'s at t'.	Bearings at r'.	Diff. of Alt.
	° ′	° ′	
1	3 11 E	9 4	41 A
2	4 27	16 21	97
3	7 15	21 33	205
4	9 12	27 0	331
5	10 30	29 15	414
6	11 18	31 14	480
7	11 26	35 39	569
8	12 21	38 50	686
9	12 42	41 44	777
10	13 18	44 21	887
11	13 32	46 36	971

SECTION 57.

Pole	Vert. ∠'s at k.	Bearings at t'.	Diff. of Alt.
	° ′	° ′	
1	7 51 E	29 43	176 A
2	11 24	34 13	299
3	13 22	37 54	397
4	14 52	42 20	509
5	16 40	47 44	675
6	16 46	50 15	833

SECTION 58.

Pole	Vert. ∠'s at k.	Bearings at t'.	Diff. of Alt.
	° ′	° ′	
1	0 3 E	5 28	5 A
2	6 10	8 5	44
3	10 3	13 52	121
4	10 20	29 37	313
5	14 20	32 18	497
6	15 0	33 18	547

SECTION 59.

Pole	Vert. ∠'s at M'.	Bearings at h.	Diff. of Alt.
	° ′	° ′	
1	28 16 D	Not seen.	
2	25 20	66 56	910 D
3	22 23	74 50	1056
4	22 12	76 20	1114
5	21 40	79 10	1232
6	21 20	83 32	1521
7	20 14	85 50	1656
8	19 12	87 30	1758
9	18 46	89 17	1978
10	17 30	90 20	2016
11	16 8	91 5	1990
12	14 42	92 27	2093
13	13 35	92 54	2035
14	12 46	93 14	1991
15	11 30	93 45	1915
16	10 30	94 7	1845

The following are the irregular sections. In the first column is the number of poles; in the second the vertical angles; in the third and fourth the two bearings, or horizontal angles, at each end of the base; and in the fifth the computed result, being the difference of altitude between the foot of each pole, and the point mentioned in the second column, where the vertical angles were taken.

SECTION 60.

Pole	Vert. ∠'s at H.	Bear. at H.	Bear. at G.	Diff. of Alt.
1	8° 30' E	70° 55'	66° 39'	832 A
2	8 30	66 46	70 33	850
3	8 30	62 56	75 9	884
4	8 30	58 20	79 20	892
5	8 30	54 27	82 42	891

SECTION 61.

Pole	Vert. ∠'s at H.	Bear. at H.	Bear. at G.	Diff. of Alt.
1	9° 31' E	68° 51'	71° 0'	1004 A
2	9 31	64 19	77 50	1091
3	9 31	60 19	80 38	1072
4	9 31	55 40	84 34	1066

SECTION 62.

Pole	Vert. ∠'s at H.	Bear. at H.	Bear. at G.	Diff. of Alt.
1	11° 0' E	72° 45'	72° 40'	1334 A
2	11 0	69 57	76 21	1389
3	11 0	62 40	82 29	1376
4	11 0	59 7	84 22	1327

SECTION 63.

Pole	Vert. ∠'s at H.	Bear. at H.	Bear. at G.	Diff. of Alt.
1	12° 15' E	74° 9'	74° 0'	1613 A
2	12 15	70 14	78 7	1652
3	12 15	67 32	80 52	1669
4	12 15	64 2	84 1	1664

SECTION 64.

Pole	Vert. ∠'s at W.	Bear. at W.	Bear. at L.	Diff. of Alt.
1	5° 57' E	74° 37'	81° 12'	382 A
2	6 57	78 12	76 36	333
3	7 11	81 12	73 19	329
4	6 19	86 33	68 34	394
5	7 5	92 16	64 4	334
6	7 48	96 43	60 49	271
7	8 40	100 10	58 40	184
8	8 52	102 42	56 41	167
9	8 41	106 0	54 23	168
10	8 30	108 53	52 16	176

SECTION 65.

Pole	Vert. ∠'s at b.	Bear. at b.	Bear. at a.	Diff. of Alt.
1	2° 1' D	47° 47'	19° 56'	20 D
2	1 15 E	32 1	26 45	26 A
3	2 59	27 0	32 56	65
4	4 35	23 40	45 18	118
5	5 0	19 2	63 47	150
6	5 39	15 26	98 12	202
7	6 1	13 3	121 47	238
8	5 44	9 29	143 12	247

SECTION 66.

Pole	Vert. ∠'s at d.	Bear. at d.	Bear. at c.	Diff. of Alt.
1	14° 25' D	162° 37'	11° 30'	870 D
2	15 55	158 15	13 36	823
3	16 57	152 52	16 35	827
4	16 12	149 43	18 0	831
5	20 57	147 3	18 41	866
6	23 5	136 18	23 37	865
7	23 40	129 0	27 25	876
8	23 3	122 16	31 32	877
9	23 46	112 59	36 25	894
10	23 23	104 22	41 34	891
11	23 35	91 13	48 53	892
12	21 21	82 49	57 38	902
13	20 11	77 38	63 0	897
14	18 47	69 33	71 24	890
15	17 31	65 26	77 38	893
16	16 1	61 51	88 57	884
17	14 33	59 1	89 55	875
18	12 48	56 49	96 15	868
19	11 38	54 45	100 45	850

SECTION 67.

Pole	Vert. ∠'s at P.	Bear. at P.	Bear. at H.	Diff. of Alt.
1	5° 42' D	47° 54'	61° 12'	476 D
2	7 42	54 10	53 56	592
3	7 13	55 23	52 22	549
4	7 12	58 40	50 36	532
5	8 1	63 52	46 43	564
6	6 47	68 29	44 48	469
7	5 27	71 25	43 18	369
8	5 31	79 14	40 16	368
9	4 15	81 17	39 11	278
10	3 51	87 37	36 34	247
11	3 48	90 58	35 16	242
12	1 5	96 47	32 56	64
13	1 27	96 53	30 46	80

	° ′	° ′	° ′	
14	3 42	91 16	32 0	208
15	6 0	83 54	34 35	348
16	7 21	69 39	38 26	435
17	9 5	64 1	41 18	564
18	9 34	55 56	44 56	625
19	10 1	48 45	49 59	706
20	9 24	41 45	54 41	703
21	9 52	37 19	59 16	777
22	10 0	32 45	65 51	841
23	9 28	29 27	71 0	819

SECTION 68.

Pole	Vert. ∠'s at P.	Bear. at P.	Bear. at G.	Diff. of Alt.
	° ′	° ′	° ′	
1	6 23 D	21 51		
2	6 20	22 44	139 42	140 D
3	6 20	24 2	136 17	1092
4	6 14	25 20	132 35	1025
5	5 54	27 20	127 52	932
6	5 41	29 23	121 57	843
7	5 21	32 27	115 58	769
8	5 29	35 59	107 50	740
9	5 45	41 31	87 8	615
	° ′	° ′	° ′	
10	5 30	45 10	82 44	578
11	5 42	49 50	77 53	589
12	5 56	53 57	74 41	613
13	6 3	58 6	70 2	605
14	5 4	64 0	66 29	510
15	5 8	67 27	65 5	527
16	4 11	73 46	65 27	486
17	4 20	76 52	64 11	518
18	4 21	80 26	62 58	542

SECTION 69.

Pole	Vert. ∠'s at H.	Bear. at H.	Bear. at W.	Diff. of Alt.
	° ′	° ′	° ′	
1	12 47 E	116 58	51 34	1809 A
2	14 36	107 42	59 25	2135
3	13 56	101 11	65 9	2023
4	13 43	96 35	69 7	1960
5	13 9	92 53	72 30	1875
6	12 38	88 41	76 6	1760
7	12 28	82 53	81 18	1703
8	12 8	78 24	85 45	1656
9	11 47	76 54	87 30	1646
10	11 26	74 56	89 25	1592
11	10 6	68 32	96 23	1452

The three following sections were taken in a manner different from all the rest. They were made by measuring in a straight sloping line, or nearly straight, from certain points, towards K and N; and, at the beginning of the line, taking the angle of elevation or depression of several places or points in it, whose distance from the beginning were measured. In these cases, each distance is the hypothenuse of a right-angled triangle; and the manner of operation is this, as radius is to the hypothenuse, or measured slope distance, so is the sine of the elevation or depression, to the difference of altitude, and so is the cosine of the same vertical angle, to the horizontal distance.

SECTION 70, from M′ to K.

Pole	Slope Dist.	Vert. ∠'s at M′.	Horiz. Dist.	Diff. of Alt.
		° ′		
1	463	7 50½ D	459	60
2	794	6 54½	788	92
3	992	6 50	985	114
		° ′		
4	1257	5 28½	1251	116
5	1455	5 25½	1449	134
6	1824	5 4	1817	157
	Ends at K.			

Pole	Slope Dist.	Vert. ∠'s at ♉.	Horiz. Dist.	Diff. of Alt.	Pole	Slope Dist.	Vert. ∠'s at ♉.	Horiz. Dist.	Diff. of Alt.
SECTION 71, from ♉ to K.					SECTION 72, from ♉ to N.				
1	727	13° 59′½ D	706	167	1	528	3° 10′ E	527	38
2	1455	9 44½	1434	242	2	1188	4 10	1185	95
3	1720	9 4½	1699	267	3	1594	6 8	1585	179
4	1984	7 52	1965	267		Ends at N.			
5	2547	7 3	2528	308					
	Ends at K.								

Besides these sections, there were many more single points, whose places and relative altitudes were observed, and computed; but it is not necessary to abstract them all here.

The following plate, N°. 1, has 72 figures, answering to these 72 sections, each to each, according to the numbers. In these figures, the line having the letters *p*, *p*, *p*, &c, annexed, is the section line, the letters *p*, *p*, &c, denoting the poles; the other line, forming the angle with the section, is the base line; and between them are the degrees and minutes contained in the angle formed by them; at the angular point was observed the elevation or depression of each point *p*; and the bearings or horizontal angles were observed at the other end of the base, whence faint lines are drawn to some of the points *p*, forming with the base line those horizontal angles. The base and section lines, in each figure, are also drawn nearly in the same direction as they are in the plan, or on the ground, supposing the top of the paper to represent the North, towards which a person looks when viewing the ground from the South.

Having finished the computation of the relative altitudes of all the points, in the sections, the next consideration is, how they are to be applied in determining the attraction of the hill. In whatever manner this last mentioned operation may be performed, it is evident, that all the points and sections, with their altitudes, must be entered in the plan. Therefore, having accurately constructed a large plan of the ground, as before mentioned, containing all the principal lines or bases, at the extremities of which either vertical or horizontal angles were

taken, from them are then determined, in this plan, the places of all the other points in the sections, whether vertical, horizontal, or inclined. These places or points were determined, by drawing lines from each extremity of the base, so as to form with it angles equal to those which were observed on the ground, for each corresponding pole: the intersections of these lines are the places of the poles, which having marked with a fine dot or point of ink, and written close to each point the proper number expressing its relative altitude, and cleaned the paper by rubbing out the lines forming the angles, by which the points were determined, there remained only the points, with the figures expressing their altitudes, distinctly exhibited in the plan. See plate 2.

It remains now to apply all the foregoing calculations, and constructions, to determine the effect of the attraction, in the direction of the meridian. And here it soon occurs, that the best method is, to divide the plan into a great number of small parts, which may be considered as the bases of as many vertical columns, or pillars of matter, into which the hill, and the adjacent ground, may be supposed to be divided, by vertical planes, forming an imaginary group of vertical columns, something like a set of basaltine pillars, or like the cells in a piece of honeycomb; then to compute the attraction of each pillar separately, in the direction of the meridian; and lastly, to take the sum of all these computed effects, for the whole attraction of the matter in the hill, &c. Now the attraction of any one of these pillars, on a body in a given place, may be easily determined, and that in any direction, to a sufficient degree of accuracy, because of the smallness and given position of the base: for, on account of its smallness, all the matter in the pillar may be supposed to be collected into its axis, or vertical line erected on the middle of the base; the length of which axis, as the mean altitude of the pillar, is to be estimated from the altitudes of the points in the plan which fall within, and near, the base of the pillar: then, having obtained the altitude of this axis, with the position of its base, and the matter supposed to be collected into it, a theorem can easily

be assigned, by which the effect of its attraction may be computed. But, to retain the proper degree of accuracy in this computation, it is evident that the plan must be divided into a great number of parts, perhaps not less than a thousand for each observatory, in order that they may be sufficiently small, and by this means forming about 2000 of such pillars of matter, whose attractions must be separately computed, as above mentioned. The labour and time necessary for such computation, it is evident, would be very great, perhaps not less than those employed in all the preceding computations of the sections, and all the other points and lines concerned in this business. For this reason, I was desirous of obtaining a theorem, or method, by which the attractions of the small and numerous pillars might be computed, with the same degree of accuracy, but with less expence of labour and time, than when computed separately as above mentioned. And in this inquiry the success has been equal to my wishes, having at length devised a method, by which the business has been effected in perhaps one-fourth or one-fifth of the time, that would have been required in the other way.

Of all the methods of dividing the plan into a great number of small parts, I have found that to be the most convenient for the computation, in which it is first divided into a number of rings by concentric circles, and these again divided into a sufficient number of parts by radii drawn from the common centre, that centre being the observatory where the plummet is placed, on which the effect of attraction is to be computed. By this means, the plan is divided into a great number of small quadrilateral spaces, two of the opposite sides of which are small portions of adjacent circles, and the other two are the intercepted small parts of two adjacent radii, as appears by fig. 1, pl. 3, in which, for the present, let the circles and their radii be supposed to be drawn at any distances whatever from each other, till it shall appear, from the theorem to be investigated, what may be the most convenient distances and positions of those lines. In this figure, A is the observatory, AN the meridian, WAE an east-and-west line, BCDE one of the

small spaces, and F its centre, or the foot of the axis of the pillar whose base is BCDE; the figure AWNEA being a horizontal or level section, through the point A. Join A, F, and with the centre A describe the arc of the mid circle GFH. Let a denote the length of the axis on the point F, or the mean height of the pillar on the base BD; and s the sine of the angle of elevation of that pillar, as observed at A, to the radius 1, or $s = \frac{a}{\sqrt{(a^2 + AF^2)}}$. Then will the magnitude of that column, or its quantity of matter, be expressed by $(BC + DE) \times \frac{1}{2}BE \times a$, which is supposed to be all collected into the axis: consequently, if the attraction of each particle of matter be in the reciprocal duplicate ratio of its distance, the attraction of the matter in the pillar so placed, on the plummet at A, in the direction of the meridian AN, will be

$$\frac{BC+DE}{2AF} \times BE \times a \times \frac{s}{a} \times c = \frac{BC+DE}{2AF} \times BE \times sc = \frac{GH}{AF} \times BE \times sc$$

nearly, supposing F to be equally distant from BC and DE; and c the cosine of the angle FAN, to the radius 1.

But $\frac{GH}{AF} \times c$ is nearly equal to d, the difference of the sines of the angles BAN, CAN, as is thus demonstrated. Draw GK, FL, HM, perpendicular to AW, and GP parallel to the same; also draw the chord GH. Then AK, AM are the sines of the angles GAN, HAN, to the radius AF, their difference being KM = GP; also FL is the cosine of FAN, to the same radius: consequently GP : FL :: d : c. But the triangles LFA, PGH are equiangular, and therefore GP : FL :: GH : AF. Consequently GH : AF :: $d : c$; or $\frac{GH}{AF} \times c = d$. This equation is accurately true, when GH is the chord of the arc; and, as the small arc differs insensibly from its chord, the same equation is sufficiently near the truth when GH is the arc itself. Substituting now d instead of the quantity $\frac{GH}{AF} \times c$, in the theorem above, it will become $BE \times ds$ for the measure of the attraction of the pillar whose base is BD, in the direction AN. Which is as easy and simple an expression for the attraction of a single pillar, as can well be desired or expected.

But to make the application of this theorem still more easy,

to the great number of small pillars concerned in this business, let us suppose BE and d to be constant or invariable quantities; and then it is evident that we shall have nothing more to do, but to collect all the s's, or sines of elevation, of all the pillars, into one sum, and then multiply that sum by the constant quantity BE $\times d$, by which there will be produced the measure of the attraction of all the pillars, or of the whole part of the ground on one side of WE. Now BE will be made to become constant, by making the circles equidistant from one another, or by taking the radii in arithmetical progression. And d will be constant, by drawing the radii so as to form, with AN, angles whose sines shall be in arithmetical progression; for then d is the common difference of the sines of those angles. Hence then we are easily led to the best manner of dividing the plan into the small spaces, viz, from the centre A describe a sufficient number of concentric and equi-distant circles; divide the radius AI, of any one of them, into a sufficient number of equal parts, and from the points of division erect perpendiculars to meet the circle; then through the points of intersection draw radii, and they will divide the circles in the manner required.

In a computation of this kind, we need only calculate the attraction of the matter which is above the plane or horizon of each observatory, and the attraction of so much matter as is wanting, to fill up the vacuity below that plane, lying between it and the surface of the lower part of the hill. For the South observatory, the attraction of the Southern parts that are above it, must be subtracted from that of the Northern parts, to obtain the attraction of the whole towards the North; that is, the Southern elevations are negative, and the Northern ones affirmative. The contrary names take place with respect to the depressions, or the vacuities below the plane of the observatory; for if the whole space below this horizontal plane were full of matter, to an equal extent both ways, its attraction need not be computed, as those on the contrary sides would mutually balance each other; but since there are unequal vacuities on each side, it is evident, that

the attraction of the matter that might be contained in them, must be deducted from the other two equal quantities, to leave the real attraction of those two sides; then subtracting the remainder on the South side, from that of the Northern side, there will at last remain the joint effect of all the matter below the plane, in the Northern direction: but as the one remainder is to be subtracted from the other, the two equal quantities may be omitted in both, and only the effects of the vacuities brought into the account, which being twice subtracted, their signs become contrary to those of the parts above the horizontal plane; that is, the effect of the Southern vacuity is affirmative, and that of the Northern one negative. But for the Northern observatory, when the attraction towards the South is to be found, the contrary names take place; that is, in the elevations the Southern parts are affirmative, and the Northern parts negative; but in the vacuities or depressions, the Northern parts are affirmative, and the Southern ones negative.

According to the foregoing method the plan of the ground was divided into 20 rings, by equidistant concentric circles, described about each observatory as a centre; and each quadrant was divided into 12 parts, or sectors, by lines forming, with the meridian, angles whose sines were in arithmetical progression; by which means, the space in each quadrant was divided into 240 small parts, making almost a thousand of such parts in the whole round, for each observatory, or near 2000 for the two observatories. This was judged to be a sufficiently great number of parts, to afford a very considerable degree of accuracy; or at least that number was as great, and the parts as small, as was well consistent with the degree of accuracy afforded by the number of points, whose relative altitudes had been determined.

In this division, the common breadth of the rings, or the common difference of the radii, is $666\frac{2}{3}$ feet; and the common difference of the sines of the angles, formed by the radii and the meridian, is $\frac{1}{12}$th of the radius; and consequently, those angles are expressed, in degrees and minutes, as here below,

viz, 4° 47′, 9° 36′, 14° 29′, 19° 28′, 24° 37′, 30° 0′, 35° 41′, 41° 48′½, 48° 35′, 56° 26′½, 66° 26′½, 90° 0′.

Plate 2 contains a small plan of the principal and most central part of the ground, accurately divided in the above manner, for one of the observatories, namely, the Northern one, with the places of all, or most of the points, which fall within this part of the ground, accurately laid down, and marked with dots, as also such of the included letters as have been before mentioned in this paper.

In this plate RABCD &c, is the chain of stations, around the hill; K and N are the East and West cairns, on the extremities of the ridge of the hill; O the Southern observatory, and P the Northern one. Of this kind two large plans were made, one divided for each observatory, from which were estimated the mean altitudes of the pillars, erected on the spaces into which they were divided.

These altitudes are easily estimated, when several of the points fall near and in the small spaces or bases, especially when they are near the middle of them; but, numerous as the points are, there were many bases in which none at all were contained, nor even near them. This circumstance at first gave much trouble and dissatisfaction, till I fell upon the following method, by which the defect was in a great measure supplied, and by which I was enabled to proceed in the estimation of the altitudes, both with much expedition, and a considerable degree of accuracy. This method was, the connecting together, by a faint line, all the points which were of the same relative altitude: by so doing, I obtained a great number of irregular polygons, lying within, and at some distance from, one another, and bearing a considerable degree of resemblance to each other: these polygons were the figures of so many level or horizontal sections of the hills, the relative altitudes of all the parts of them being known; and as every base, or little space, had several of them passing through it, I was thus able to determine the altitude belonging to each space with much ease and accuracy. In this estimation, I could generally be pretty sure of the altitude to within 10

feet, and often within 5, which on an average might be about the 100th part of the whole altitude; and when we consider that the number of such estimated altitudes is very great, and that it is probable the small errors among them would nearly balance one another, the defect of those that might be reckoned too little, being compensated by the excess in those which might be taken too great, we need not hesitate to pronounce, that the error arising from the estimation of the altitudes, is probably still much less than that part.

It was necessary to determine these altitudes of the pillars, in order to compute the sines of the angles of elevation subtended by them, as the theorem requires the use of these sines; and the very easy method used in deducing the latter from the former, is explained after registering the altitudes of all the pillars, as they were computed. This register consists of 16 tables, namely, 4 quadrants of spaces in the altitudes, and 4 in the depressions, for each observatory, as specified in the titles of them. The numbers are feet, like all the other dimensions. The numbers on the same horizontal line, from left to right, are such as are all in the same ring; and those in one and the same vertical column, are in the same sector, or between the same two radii; the number of the ring, counted from the common centre, is written in the left-hand margin; and the number of the vertical column, or distance of the space, or sector, from the meridian, at the top; also the radius of each ring, that is, the line from the common centre to the middle of the ring, is written on the same line with it, in the right-hand margin. It may be further remarked, that in such little spaces as were cut through by the boundary line, between elevations and depressions, thus making but a part of such spaces in each of those denominations, each space was accounted as a whole one; but then the mean altitude or depression, in each part, was diminished in the proportion of the whole space, to the part of it so included in the boundary. The altitudes and depressions are set down first with respect to the Southern observatory O; then for the Northern observatory P; and, in each, the altitudes are placed first.

1. Altitudes above o in the N.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	215	215	215	215	210	205	200	190	170	145	120	75	333⅓
2	605	610	605	600	595	590	580	570	510	450	350	200	1000
3	965	1005	1010	1010	1020	1050	1040	900	810	600	415	220	1667
4	670	680	700	780	860	930	1040	1090	1100	760	480	210	2333
5	280	310	370	450	560	700	830	960	1180	890	545	200	3000
6	20	50	100	110	250	380	525	710	890	950	605	110	3667
7	.	.	.	.	10	70	120	415	620	780	600	120	4333
8	.	.	.	.	.	.	15	95	390	610	480	35	5000
9	.	.	.	.	.	.	.	.	135	310	220	5	5667
10	.	.	.	.	.	.	.	.	.	40	20	.	6333

2. Altitudes above o in the N.E. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	210	205	205	200	195	185	170	155	140	125	105	70	333⅓
2	550	545	540	530	520	510	500	465	430	370	270	130	1000
3	910	840	825	815	800	760	720	680	635	590	500	200	1667
4	645	640	635	640	645	650	675	715	730	700	580	300	2333
5	265	255	265	285	310	350	390	450	460	500	600	280	3000
6	10	12	20	65	100	130	160	180	180	320	460	300	3667
7	.	.	.	.	.	.	.	15	55	110	150	250	4333
8	.	.	.	.	.	.	.	.	.	.	.	50	5000

3. Altitudes above o in the S.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	.	.	.	.	.	.	.	.	.	.	.	10	333⅓
2	.	.	.	.	.	.	.	.	.	.	.	15	1000
3	.	.	.	.	.	.	.	.	.	.	.	5	1667
12	.	.	.	.	.	.	.	.	.	.	.	40	7667
13	.	.	.	.	.	.	.	.	.	.	.	60	8333
14	.	.	.	.	.	.	.	.	.	.	.	100	9000
15	.	.	.	.	.	.	.	.	.	.	.	200	9667
16	40	120	200	250	280	280	260	170	60	.	.	300	10333
17	160	270	360	400	440	450	450	390	270	30	.	500	11000
18	310	420	500	580	620	650	660	630	500	200	.	700	11667
19	440	540	620	680	740	800	800	780	600	400	60	800	12333
20	550	650	750	800	900	950	960	960	800	500	200	900	13000

4. Altitudes above o in the S.E. quarter.

Rings	1	2	3	4	5	6	7	8	.	.	.	.	Radii.
16	10	.	.	.	.	.	.	.	.	.	.	.	10333
17	80	60	50	50	10	.	.	.	.	.	.	.	11000
18	220	200	180	130	70	30	.	.	.	.	.	.	11667
19	340	300	260	240	170	120	20	.	.	.	.	.	12333
20	450	410	380	330	260	180	100	20	.	.	.	.	13000

5. Depressions below o in the N.W, quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
6	70	40	15	5	.	.	.	.	.	.	.	15	3667
7	250	240	200	150	60	30	.	.	.	.	.	40	4333
8	460	450	430	390	280	200	80	10	.	.	.	80	5000
9	700	700	680	630	520	450	340	170	15	.	.	120	5667
10	840	830	800	780	650	600	520	380	180	40	70	220	6333
11	960	920	880	850	750	650	630	550	350	250	300	430	7000
12	1100	1000	950	900	820	780	780	780	580	530	500	560	7667
13	1130	1080	980	880	840	800	830	860	780	690	630	640	8333
14	1180	1100	1000	900	900	900	910	940	870	800	700	500	9000
15	1180	1100	1100	1080	1040	1050	1060	1070	1000	870	730	300	9667
16	1100	1100	1100	1100	1100	1140	1150	1150	1120	990	760	160	10333
17	1100	1100	1100	1130	1180	1200	1200	1200	1180	1080	700	80	11000
18	1100	1100	1150	1200	1200	1150	1100	1100	1200	1180	700	100	11667
19	1100	1120	1220	1230	1260	1200	1200	1200	1300	1240	620	60	12333
20	1120	1220	1320	1360	1390	1390	1390	1340	1440	1300	620	50	13000

6. Depressions below o in the N.E. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
6	70	60	30	10	.	.	.	.	.	.	.	10	3667
7	260	240	200	150	110	80	30	40	30	10	.	10	4333
8	450	440	400	350	280	180	100	180	180	190	190	40	5000
9	700	690	680	610	520	400	200	240	290	350	330	200	5667
10	850	870	890	860	770	620	440	300	380	500	450	370	6333
11	1020	1060	1070	1050	980	860	700	520	400	650	600	530	7000
12	1140	1160	1180	1160	1140	1080	950	840	620	720	850	700	7667
13	1200	1190	1200	1220	1240	1250	1160	1050	900	840	950	880	8333
14	1230	1130	1050	1050	1100	1220	1260	1220	1070	950	1020	990	9000
15	1100	960	900	850	900	1100	1230	1210	1170	1060	1090	1100	9667
16	970	860	880	780	780	900	1120	1180	1200	1180	1160	1150	10333
17	970	800	760	750	750	780	1000	1200	1300	1240	1200	1100	11000

7. Depressions below o in the S.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	165	165	160	155	150	140	130	120	110	90	50	10	333⅓
2	400	390	380	370	350	330	300	270	240	210	160	60	1000
3	600	580	560	530	500	570	540	400	370	340	280	100	1667
4	740	720	700	670	640	610	580	530	490	440	370	160	2333
5	800	800	800	770	740	710	660	610	570	510	440	230	3000
6	780	790	780	770	780	800	790	700	650	590	510	320	3667
7	700	710	720	730	750	750	750	750	730	670	600	400	4333
8	580	590	600	610	640	660	700	720	730	730	760	520	5000
9	490	490	490	480	490	510	600	650	660	690	580	450	5667
10	470	460	420	400	420	420	440	490	580	590	560	430	6333
11	340	340	340	340	340	330	350	390	450	480	380	370	7000
12	210	220	230	250	250	250	280	310	340	370	250	200	7667
13	160	150	140	120	130	150	200	230	280	290	230	110	8333
14	110	90	60	20	20	20	70	150	230	240	150	90	9000
15	50	20	.	.	.	.	.	40	140	180	90	50	9667
16	.	.	.	.	.	.	.	.	30	90	70	20	10333

8. Depressions below o in the S.E. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	165	165	160	155	150	140	130	120	110	100	95	40	333⅓
2	400	400	400	400	400	390	380	360	330	300	250	110	1000
3	600	610	610	610	610	600	600	580	550	500	440	200	1667
4	760	750	740	740	740	730	720	710	680	640	560	300	2333
5	800	800	800	800	800	800	800	800	790	740	660	400	3000
6	780	780	780	780	780	790	800	840	880	850	750	470	3667
7	700	690	680	670	670	680	620	720	820	900	770	520	4333
8	580	570	570	570	570	580	590	600	660	800	800	600	5000
9	490	490	490	490	490	490	490	500	520	700	880	600	5667
10	470	460	450	440	430	420	410	430	470	530	880	680	6333
11	340	330	320	320	320	320	330	350	420	500	780	780	7000
12	210	200	200	200	210	220	240	280	390	480	680	900	7667
13	120	120	130	130	140	150	180	230	300	450	600	990	8333
14	110	110	110	120	130	150	160	200	280	440	580	980	9000
15	70	70	70	70	90	120	140	170	240	420	570	990	9667
16	10	20	30	40	50	80	120	160	220	400	550	1000	10333
17	.	.	.	.	10	40	90	140	200	340	540	950	11000
18	.	.	.	.	.	5	40	110	170	300	500	850	11667
19	.	.	.	.	.	.	.	50	150	280	470	780	12333
20	.	.	.	.	.	.	.	20	150	250	400	700	13000

9. Altitudes above P in the N.W. quarter.

Rings												Radii.
	.	.	.	.	.	.	.	.	.	.	12	
4	.	.	.	.	.	.	.	.	.	.	10	2333
5	.	.	.	.	.	.	.	.	.	.	15	3000
6	.	.	.	.	.	.	.	.	.	.	15	3667
7	.	.	.	.	.	.	.	.	.	.	15	4333
8	.	.	.	.	.	.	.	.	.	.	15	5000

10. Altitudes above P in the N.E. quarter.

Rings												Radii.
	.	.	.	.	.	.	.	.	.	.	12	
1	.	.	.	.	.	.	.	.	.	.	10	$333\frac{1}{3}$
2	.	.	.	.	.	.	.	.	.	.	10	1000
3	.	.	.	.	.	.	.	.	.	.	15	1667
4	.	.	.	.	.	.	.	.	.	.	60	2333
5	.	.	.	.	.	.	.	.	.	.	40	3000
6	.	.	.	.	.	.	.	.	.	.	5	3667

11. Altitudes above P in the S.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	110	110	105	105	100	100	95	95	90	85	80	35	$333\frac{1}{3}$
2	340	330	320	310	300	290	280	270	240	210	170	90	1000
3	660	660	660	660	660	650	620	590	570	510	370	170	1667
4	1020	1030	1040	1050	1060	1070	1030	990	910	800	660	270	2333
5	1030	1110	1280	1270	1320	1330	1310	1280	1270	1170	910	460	3000
6	670	770	810	900	930	950	1020	1070	1150	1270	1100	600	3667
7	280	340	420	480	540	570	620	670	720	880	1050	660	4333
8	20	50	90	140	210	290	350	420	490	570	700	570	5000
9	.	.	.	.	.	.	120	210	270	320	370	270	5667
10	.	.	.	.	.	.	.	.	90	150	180	20	6333
15	.	.	.	.	.	.	.	.	140	170	.	.	9667
16	.	.	.	.	.	.	.	.	140	470	30	.	10333
17	.	.	.	.	.	.	.	.	130	500	170	170	11000
18	.	.	.	.	.	.	.	20	170	600	400	400	11667
19	150	150	140	120	90	60	30	70	170	500	500	500	12333
20	160	170	210	220	200	170	100	130	170	400	600	600	13000

12. Altitudes above P in the S.E. quadrant.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	110	110	105	105	100	100	95	95	90	85	80	35	$333\frac{1}{3}$
2	340	330	320	310	300	290	280	270	240	210	170	90	1000
3	660	640	620	600	570	540	510	480	440	380	290	160	1667
4	1000	980	950	910	870	810	730	670	540	460	330	170	2333
5	1020	1020	1020	1030	1030	1020	970	770	570	470	390	130	3000
6	670	710	770	810	840	860	910	890	720	650	400	30	3667
7	290	320	360	390	470	590	700	750	780	600	280	.	4333
8	.	.	.	20	70	170	250	420	630	550	170	.	5000
9	.	.	.	.	.	.	.	110	360	420	170	.	5667
10	.	.	.	.	.	.	.	.	70	180	120	.	6333
18	.	20	40	50	40	.	.	.	.	.	.	.	11667
19	100	100	100	100	80	30	.	.	.	.	.	.	12333
20	130	130	130	120	110	80	.	.	.	.	.	.	13000

13. Depressions below P in the N.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	100	95	90	85	80	75	70	60	50	40	30	15	$333\frac{1}{3}$
2	390	380	360	330	310	290	270	240	210	180	150	60	1000
3	520	510	500	490	480	470	450	430	410	370	270	80	1667
4	650	640	620	610	590	570	550	530	500	460	390	90	2333
5	830	820	760	720	690	660	630	590	560	500	380	130	3000
6	880	860	850	790	730	700	670	640	580	480	340	280	3667
7	910	900	860	830	790	720	630	620	540	550	440	185	4333
8	930	890	840	800	830	710	610	610	580	530	520	430	5000
9	830	830	830	830	830	830	760	700	670	620	600	330	5667
10	730	740	755	770	785	800	815	830	780	750	720	460	6333
11	730	750	780	800	830	860	860	880	880	860	820	530	7000
12	750	770	810	860	910	930	950	960	950	930	880	580	7667
13	770	840	910	950	990	1030	1050	1050	1030	980	950	650	8333

14. Depressions below P in the N.E. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
1	80	75	70	65	60	55	50	45	40	35	30	20	333⅓
2	330	325	320	300	280	260	240	220	190	150	110	30	1000
3	520	515	505	490	475	460	440	420	400	330	240	60	1667
4	660	675	690	700	700	690	660	620	560	470	350	80	2333
5	840	840	840	840	840	830	820	770	720	620	440	100	3000
6	860	880	900	920	930	930	910	870	830	740	570	240	3667
7	920	920	920	880	880	900	930	940	930	840	680	610	4333
8	920	840	780	780	740	720	770	870	920	970	900	630	5000
9	720	670	600	600	600	560	580	670	850	950	940	600	5667
10	700	620	520	500	500	500	500	520	720	920	960	650	6333
11	700	600	600	600	620	600	580	560	540	840	920	770	7000
12	720	700	680	700	720	740	700	740	570	800	920	820	7667
13	720	720	720	720	700	700	720	720	620	820	900	920	8333

15. Depressions below P in the S.W. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
9	230	140	100	70	40	40	.	.	.	.	.	.	5667
10	400	340	280	230	190	150	110	30	.	.	.	.	6333
11	500	470	410	340	290	260	230	200	160	30	.	140	7000
12	500	510	510	490	410	370	330	310	350	280	150	260	7667
13	480	500	500	510	500	460	430	260	150	230	280	360	8333
14	370	390	400	430	450	450	400	210	10	110	280	230	9000
15	260	260	250	260	330	330	310	130	.	.	130	130	9667
16	200	200	150	160	170	220	230	130	.	.	.	.	10333
17	130	130	90	80	90	110	140	30	.	.	.	.	11000
18	10	20	30	10	10	20	30	.	.	.	.	.	11667

16. Depressious below P in the S.E. quarter.

Rings	1	2	3	4	5	6	7	8	9	10	11	12	Radii.
7	.	.	.	.	.	.	.	.	.	.	.	80	4333
8	20	30	80	.	.	.	.	.	.	.	.	210	5000
9	260	290	290	280	240	150	30	.	.	.	.	270	5667
10	420	440	450	440	420	370	270	140	.	.	.	330	6333
11	530	540	560	560	550	480	430	330	150	40	40	430	7000
12	500	510	520	550	630	600	500	430	290	230	200	630	7667
13	450	430	420	410	430	570	630	530	430	480	340	710	8333
14	360	330	310	290	280	330	510	670	590	630	570	830	9000
15	240	230	220	200	180	200	530	530	770	760	710	870	9667
16	180	160	150	130	110	140	230	330	630	830	790	880	10333
17	110	80	50	40	30	90	190	280	500	860	830	860	11000
18	10	.	.	.	.	10	150	260	400	760	830	760	11667
19	.	.	.	.	.	.	70	230	330	600	770	630	12333
20	.	.	.	.	.	.	10	180	290	530	690	530	13000

It remains next to find the sines of the vertical angles, subtended by all the foregoing altitudes and depressions; since the sum of these sines is the thing we are in quest of. Now, each altitude, or depression, is the perpendicular of a right-angled triangle, of which the given radius, standing on the same line with it, in the right-hand margin, is the base, or the other side about the right angle; and by the resolution of the right-angled triangle, for each perpendicular, the same number of corresponding sines will be found. But with such data, the tangent of the angle is much easier to be found, than the sine, and the analogy for that purpose is this, as the base : to the perpendicular :: radius 1 : the tangent, which will therefore be found, by barely dividing the given perpendicular by the base; and if we find this number in its proper column, in a table of sines and tangents, then on the same line with it, in the column of sines, will be found the sine of the angle required. This seems to be the easiest way of resolving all the triangles, when computed separately. But as the labour would be very great, in performing so many hundreds of arithmatical divisions, &c, either by logarithms, or

by the natural numbers, instead of it, the following method was adopted, being a much more expeditious way of obtaining the sum of the sines required. This method consists in finding, in a very easy manner, the difference between each tangent and its corresponding sine, from the given base and perpendicular, and then, subtracting the sum of all the differences from the sum of the tangents, there remains the sum of the sines. Several advantages attend this method of proceeding: for, to find the tangents, we need not divide every perpendicular separately by its corresponding base, but add together all the perpendiculars that are on the same line, and divide their sum by the common base, which is the radius of the middle of the ring, and is placed on the same line with them towards the right-hand; for thus we shall have little more than a twelfth part of the number of divisions to perform: also, a great part of the tangents are so small, that they do not at all differ from their corresponding sines, in the number of decimals that it is necessary to continue the computations to; in all which cases therefore the trouble of finding the differences is saved; and those differences, which it is necessary to compute, are very readily found by inspection, on a peculiar kind of sliding rule, which was constructed for this purpose, and of which I shall here give a short description.

This rule (the figure of which is represented pl. 3, fig. 2) consists of three columns; one marked AF or base, which is moveable by sliding it up or down by the side of the other two, which are fixed; of these two, the one contains the perpendicular altitudes or depressions, and the other the differences between the sines and tangents, to the radius 1. To construct the numbers on this rule, form a series of logarithmic tangents in arithmetical progression, of which the first term is 9·000, and the common difference ·025; take out from a table the corresponding natural tangents, and place them in the first and second columns of base and perpendicular, and the difference between the natural sine and natural tangent in the last column, marked Diff. To make use of this

scale; look out any base, and its corresponding perpendicular, in their proper columns, that is, any radius and its corresponding altitude or depression, in the sixteen foregoing tables, without regarding the number of places they contain, and bring them to correspond; then, if they consist of the same number of places, the lower index on the slider, or first column, or that answering to 1000, points to the true difference between the sine and tangent, in the last column; but if the number of places in the base exceed that in the perpendicular by 1, the upper index 100 must be used. And in this manner were computed all the differences which were necessary to be found, and placed in their proper squares, formed by the meeting of the horizontal and vertical lines, or rings and sectoral spaces, in the following set of 16 tables, which correspond to the foregoing set of 16, each to each, according to the number of them, and marked at the tops with the numbers 1, 2, 3, &c, to 12, for the sectoral spaces, and with the number of the rings on the left-hand margin. Also, in the column immediately after the number of the ring, are placed the radii which formed the last column in the preceding tables; then, in the third column, are placed the sums of the altitudes and depressions, found in each line of the former tables; and, in the next column, the quotients found by dividing the numbers in the third column by those in the second; these quotients are the sums of the tangents belonging to each line or ring; which being all added together, their total is placed at the bottom of the column: after this follow the 12 columns of differences before mentioned, which are succeeded by one more column, containing the sums of each line of these differences which sums being added together, their total is placed at the bottom of them; and this total is the sum of all the differences between the sines and the tangents, and it is therefore subtracted from the total of the tangents in the fourth column, when at last there remains the sum of the sines as required.

1. For the sum of the sines of alt. above 0 in the N.W. quarter.

Rings	Radii.	Sum of Perps.	3÷2= Sum of Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
1	333⅓	2175	6·525	103	103	103	103	97	97	86	75	55	36	20	5	·883
2	1000	6265	6·265	88	90	88	85	84	82	78	75	56	40	20	4	790
3	1667	10045	6·027	79	86	87	87	88	97	93	64	48	21	7	1	758
4	2333	9300	3·986	12	12	13	17	23	27	39	43	45	16	4	.	251
5	3000	7275	2·425	.	1	1	2	3	6	11	16	27	13	3	.	83
6	3667	4695	1·280	.	.	.	.	.	1	1	3	7	9	2	.	23
7	4333	2735	0·631	.	.	.	.	.	.	.	1	1	3	1	.	6
8	5000	1625	0·325	.	.	.	.	.	.	.	.	.	1	1	.	2
9	5667	670	0·118	.	.	.	.	.	.	.	.	.	.	.	.	.
10	6333	60	0·009	.	.	.	.	.	.	.	.	.	.	.	.	.

27·591 = sum of tangents. Sum. of diffs. 2·795
2·796 = sum of the diffs.
24·795 = sum of the sines of alt. above 0 in the N.W. quarter.

2. For the sum of the sines above 0 in the N.E. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
1	333⅓	1965	5·895	97	92	92	87	80	71	56	43	33	24	15	5	·695
2	1000	5360	5·360	68	67	65	62	59	56	53	43	35	23	9	1	541
3	1667	8275	4·965	66	53	51	49	47	41	36	31	25	20	13	1	433
4	2333	7555	3·238	10	10	10	10	10	11	12	14	15	13	8	1	124
5	3000	4410	1·470	.	.	.	.	1	1	1	2	2	2	4	.	13
6	3667	1937	·528	.	.	.	.	.	.	.	.	.	1	1	.	2
7	4333	580	·134	.	.	.	.	.	.	.	.	.	.	.	.	.
8	5000	50	·010	.	.	.	.	.	.	.	.	.	.	.	.	.

21·600 = sum of the tangents. 1·808
1·808 = sum of the differences.
19·792 = sum of the sines.

3. For the sum of the sines above o in the S. W. quarter.

Rings	Radii.	Sum of Perps.	3÷2= Sum of Tangs.	Sum of Diffs.
1	333½	10	0·030	
2	1000	15	15	
3	1667	5	3	
12	7667	40	5	
13	8333	60	7	
14	9000	100	11	
15	9667	200	21	
16	10333	1960	190	
17	11000	3720	338	
18	11667	5770	495	
19	12333	7260	573	
20	13000	8920	686	
			2·374 = sum	

of the tangents, or sum of the sines, as the diff. between them are nothing in this quadrant.

4. For the sum of the sines above o in the S. E. quarter.

Rings	Radii.	Sum of Perps.	3÷2= Sum of Tangs.	Sum of Diffs.
16	10333	10	0·001	
17	11000	230	21	
18	11667	830	71	
19	12333	1450	118	
20	13000	2130	164	
			0·375 = sum	

of the tangents, or of the sines, the diff. being nothing.

5. For the sines below o in the N. W. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
6	3667	145	0·040	.	.	.	.	.	.	.	.	.	.	.	.	.
7	4333	970	224	.	.	.	.	.	.	.	.	.	.	.	.	.
8	5000	2370	474	.	.	.	.	.	.	.	.	.	.	.	.	.
9	5667	4325	763	1	1	1	1	1	.	.	.	.	.	.	.	5
10	6333	5910	933	1	1	1	1	1	1	.	.	.	.	.	.	6
11	7000	7520	1·074	1	1	1	1	1	1	1	.	.	.	.	.	7
12	7667	9280	1·210	2	1	1	1	1	1	1	1	.	.	.	.	9
13	8333	10140	1·219	2	1	1	1	1	1	1	1	1	.	.	.	10
14	9000	10700	1·078	1	1	1	1	1	1	1	1	1	.	.	.	9
15	9667	11580	1·198	1	1	1	1	1	1	1	1	1	1	.	.	10
16	10333	11970	1·159	1	1	1	1	1	1	1	1	1	.	.	.	9
17	11000	12250	1·105	1	1	1	1	1	1	1	1	1	1	.	.	10
18	11667	12280	1·052	1	1	1	1	1	1	1	1	1	1	.	.	10
19	12333	12750	1·034	.	.	1	1	1	1	1	1	1	1	.	.	8
20	13000	13940	1·072	.	.	1	1	1	1	1	1	1	1	.	.	8
			13·635 = sum of the tangents.													·101
			·101 = sum of the differences.													
			13·534 = sum of the sines.													

6. For the sum of the sines below o in the N.E. quarter.

Rings	Radii.	Sum of Perps.	3 ÷ 2 = Sum of Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
6	3667	180	0·049	.	.	.	.	.	.	.	.	.	.	.	.	.
7	4333	1160	268	.	.	.	.	.	.	.	.	.	.	.	.	.
8	5000	2980	596	.	.	.	.	.	.	.	.	.	.	.	.	.
9	5667	5210	919	1	1	1	1	1	.	.	.	.	.	.	.	5
10	6333	7300	1·153	1	1	1	1	1	1	.	.	.	.	.	.	6
11	7000	9440	1·349	1	2	2	2	1	1	1	.	.	.	.	.	10
12	7667	11460	1·495	2	2	2	2	2	1	1	1	.	1	1	.	15
13	8333	13080	1·570	1	1	1	1	2	2	1	1	1	1	1	1	14
14	9000	13290	1·477	1	1	1	1	1	1	1	1	1	1	1	1	12
15	9667	12670	1·311	1	1	1	.	1	1	1	1	1	1	1	1	11
16	10333	12160	1·177	.	1	.	1	.	1	.	1	.	1	.	1	6
17	11000	11850	1·077	.	.	1	.	.	1	.	1	1	.	1	1	6
			12·441 = sum of the tangents.													·085
			·085 = sum of the differences.													
			12·356 = sum of the sines.													

7. For the sum of the sines below o in the S.W. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
1	383⅓	1445	4·335	51	51	47	43	40	35	27	21	17	10	2	.	·344
2	1000	3460	3·460	28	27	26	25	19	17	13	9	6	5	2	.	·177
3	1667	5370	3·222	21	19	17	15	13	18	16	6	5	4	2	.	136
4	2333	6650	2·850	15	14	13	12	10	8	8	5	5	3	2	.	95
5	3000	7640	2·547	9	9	9	8	6	6	5	4	3	3	2	.	64
6	3667	8260	2·253	5	5	5	5	5	5	5	3	3	2	1	.	44
7	4333	8260	1·906	2	2	2	2	3	3	3	3	2	2	1	.	25
8	5000	7840	1·568	1	1	1	1	1	1	1	1	2	2	2	1	15
9	5667	6580	1·161	1	1	1	1	1	1	1	1	1	1	1	.	11
10	6333	5680	697	1	.	.	.	.	.	.	1	1	1	1	.	5
11	7000	4450	636	.	.	.	1	.	.	.	1	.	.	.	1	3
12	7667	3160	412	.	.	.	.	.	1	.	.	.	.	.	1	2
13	8333	2190	263	.	.	.	.	.	.	.	.	.	1	.	.	1
14	9000	1250	139	.	.	.	.	.	.	.	.	.	.	.	.	.
15	9667	570	59	.	.	.	.	.	.	.	.	.	.	.	.	.
16	10333	210	20	.	.	.	.	.	.	.	.	.	.	.	.	.
			25·728 = sum of the tangents.													·922
			0·922 = sum of the differences.													
			24·806 = sum of the sines.													

8. For the sum of the sines below o in the S. E. quarter.

Rings	Radii.	Sum of Deps.	3÷2= Sum o Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
1	333⅓	1510	4·530	51	51	47	43	40	35	27	21	17	13	11	1	·357
2	1000	4120	4·120	28	28	28	28	28	27	25	21	17	13	8	1	252
3	1667	6510	3·906	21	21	21	21	21	21	21	19	16	13	9	1	205
4	2333	8070	3·459	16	16	15	15	15	15	14	14	13	10	6	1	150
5	3000	8990	2·997	9	9	9	9	9	9	9	9	9	8	5	1	95
6	3667	9280	2·531	5	5	5	5	5	5	5	5	6	6	4	1	57
7	4333	8440	1·948	2	2	2	2	2	2	2	2	3	4	3	1	27
8	5000	7490	1·498	1	1	1	1	1	1	1	1	1	2	2	1	14
9	5667	6630	1·170	.	.	1	.	1	.	.	.	1	1	1	1	6
10	6333	6070	958	.	.	1	.	.	1	.	.	1	.	1	1	5
11	7000	5110	730	.	.	.	1	.	.	.	1	.	1	1	1	5
12	7667	4210	549	.	.	.	.	.	1	.	.	.	1	.	1	3
13	8333	3540	425	.	.	.	.	.	.	.	1	.	.	1	1	3
14	9000	3370	374	.	.	.	.	.	.	.	1	.	.	1	1	3
15	9667	3020	313	.	.	.	.	.	.	.	.	.	1	.	1	2
16	10333	2680	259	.	.	.	.	.	.	.	.	.	.	1	1	2
17	11000	2310	210	.	.	.	.	.	.	.	.	.	.	.	1	1
18	11667	1975	169	.	.	.	.	.	.	.	.	.	.	.	1	1
19	12333	1730	140	.	.	.	.	.	.	.	.	.	.	.	1	1
20	13000	1520	117	.	.	.	.	.	.	.	.	.	.	.	1	1
			30·403 = sum of the tangents.													1·190
			1·190 = sum of the differences.													
			29·213 = sum of the sines.													

9. For the sum of the sines above P in the N. W. quarter.

4	2333	10	·004
5	3000	15	5
6	3667	15	4
7	4333	15	3
8	5000	15	3
			0·019 = sum of

the tangents, or sum of the sines, as they have no difference in this quadrant.

10. For the sum of the sines above P in the N. E. quarter.

1	333⅓	10	·030
2	1000	10	10
3	1667	15	9
4	2333	60	26
5	3000	40	13
6	3667	5	2
			0·090 = sum of

the tangents, or sum of the sines, they being equal in this quadrant.

11. For the sum of the sines above P in the S.W. quarter.

Rings	Radii.	Sum of Alts.	3 ÷ 2 = Sum of Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
1	333⅓	1110	3·330	17	17	15	15	13	13	12	12	10	8	6	1	·139
2	1000	3150	3·150	18	17	16	14	13	12	11	10	6	5	2	1	125
3	1667	6780	4·068	28	28	28	28	28	27	23	20	18	14	5	1	248
4	2333	10930	4·684	37	38	39	40	41	42	38	34	26	18	7	1	361
5	3000	13740	4·580	18	23	34	33	38	39	37	34	33	26	14	2	331
6	3667	11240	3·065	3	5	5	6	6	7	14	16	19	21	14	2	118
7	4333	7230	1·668	.	.	1	1	1	1	1	2	2	4	5	2	20
8	5000	3900	780	.	.	.	.	.	.	1	.	1	1	1	1	5
9	5667	1560	275	.	.	.	.	.	.	.	.	.	.	1	.	1
10	6333	440	70	.	.	.	.	.	.	.	.	.	.	.	.	.
15	9667	310	32	.	.	.	.	.	.	.	.	.	.	.	.	.
16	10333	640	62	.	.	.	.	.	.	.	.	.	.	.	.	.
17	11000	970	88	.	.	.	.	.	.	.	.	.	1	.	.	1
18	11667	1590	137	.	.	.	.	.	.	.	.	.	1	.	.	1
19	12333	2480	201	.	.	.	.	.	.	.	.	.	.	.	1	1
20	13000	3130	241	.	.	.	.	.	.	.	.	.	1	.	1	2
			26·431 = sum of the tangents.													1·353
			1·353 = sum of the differences.													
			25·078 = sum of the sines.													

12. For the sum of the sines above P in the S.E. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
1	333⅓	1110	3·330	17	17	15	15	13	13	12	12	10	8	6	1	·139
2	1000	3150	3·150	18	17	16	14	13	12	11	10	6	5	2	1	125
3	1667	5890	3·534	28	26	23	21	18	16	14	12	9	5	2	1	175
4	2333	8420	3·609	34	33	30	26	24	18	15	12	6	4	1	.	203
5	3000	9440	3·147	18	18	18	18	18	18	16	9	3	2	1	.	139
6	3667	8260	2·253	3	4	5	5	5	6	8	7	4	3	1	.	51
7	4333	5530	1·276	.	.	1	.	1	1	2	3	3	1	.	.	12
8	5000	2280	456	.	.	.	.	.	.	.	1	1	1	.	.	3
9	5667	1060	187	.	.	.	.	.	.	.	.	.	1	.	.	1
10	6333	370	58	.	.	.	.	.	.	.	.	.	.	.	.	.
18	11667	150	13	.	.	.	.	.	.	.	.	.	.	.	.	.
19	12333	510	42	.	.	.	.	.	.	.	.	.	.	.	.	.
20	13000	700	54	.	.	.	.	.	.	.	.	.	.	.	.	.
			21·109 = sum of the tangents.													0·848
			0·848 = sum of the differences.													
			20·261 = sum of the sines.													

13. For the sum of the sines below P in the N.W. quarter.

Rings	Radii.	Sum of Deps.	3÷2= Sum of Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
1	333⅓	·790	2·370	13	12	9	8	7	5	5	3	2	1	1	.	·066
2	1000	3170	3·170	26	25	21	17	14	12	9	6	5	3	2	.	140
3	1667	4980	2·988	15	14	13	12	12	11	9	8	7	5	2	.	108
4	2333	6200	2·057	10	10	9	9	8	7	6	5	5	4	2	.	75
5	3000	7270	2·423	10	10	8	7	6	5	5	4	3	2	1	.	61
6	3667	7800	2·127	7	6	5	5	4	3	3	3	2	1	1	.	40
7	4333	7975	1·840	5	4	4	3	3	2	2	1	1	1	1	.	27
8	5000	8280	1·656	3	3	2	2	2	1	1	1	1	1	.	.	17
9	5667	8660	1·528	2	1	2	1	2	1	1	1	1	1	1	.	14
10	6333	8935	1·411	1	1	1	1	1	1	1	1	1	1	1	.	11
11	7000	9580	1·369	1	.	1	.	1	1	1	1	1	1	1	.	9
12	7667	10280	1·341	1	.	1	.	1	1	1	1	1	1	1	.	9
13	8333	11200	1·344	.	1	1	1	1	1	1	1	1	1	1	.	10
			26·224 = sum of the tangents.													·587
			0·587 = sum of the differences.													
			25·037 = sum of the sines.													

14. For the sum of the sines below P in the N.E. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
1	333⅓	625	1·875	7	5	5	4	3	2	2	1	1	1	1	.	·032
2	1000	2755	2·755	17	16	15	13	11	9	6	5	3	2	1	.	98
3	1667	4855	2·913	15	14	13	12	11	10	9	8	6	4	2	.	104
4	2333	7000	3·000	11	12	13	13	13	13	11	9	6	4	2	.	107
5	3000	8500	2·833	11	11	11	11	11	10	10	9	7	5	2	.	98
6	3667	9580	2·613	6	7	7	8	8	8	8	6	5	4	2	.	69
7	4333	10350	2·389	5	5	5	4	4	4	5	5	5	4	2	1	49
8	5000	11840	2·368	3	2	2	2	2	1	2	3	3	3	3	1	27
9	5667	8400	1·482	1	1	1	1	1	1	.	1	2	2	2	1	14
10	6333	7610	1·202	1	1	.	1	.	1	.	1	1	1	1	1	9
11	7000	7930	1·133	1	.	1	.	1	.	1	.	.	1	1	1	7
12	7667	8810	1·149	1	.	1	.	1	.	1	1	.	1	1	1	8
13	8333	8980	1·078	.	1	.	1	.	1	.	1	.	1	1	1	7
			26·790 = sum of the tangents.													·629
			·629 = sum of the differences.													
			[illegible]0·101 = sum of the sines.													

15. For the sum of the sines below P in the S.W. quarter.

Rings	Radii.	Sum of Deps.	3÷2= Sum of Tangs.	1	2	3	4	5	6	7	8	9	10	11	12	Sum of Diffs.
9	5667	620	·051	.	.	.	.	.	.	.	.	.	.	.	.	.
10	6333	1730	273	.	.	1	.	.	.	.	.	.	.	.	.	·001
11	7000	3030	433	1	.	.	.	.	.	1	.	.	.	.	.	2
12	7667	4470	583	.	1	.	.	.	.	.	.	1	.	.	.	2
13	8333	4600	559	.	.	.	1	.	.	.	1	.	.	.	.	2
14	9000	3730	414	.	.	.	1	.	.	.	.	.	.	1	.	2
15	9667	2390	247	.	.	.	.	.	.	1	.	.	.	.	.	1
16	10333	1460	141	.	.	.	.	.	.	.	1	.	.	.	.	1
17	11000	800	73	.	.	.	.	.	.	.	.	.	.	.	.	.
18	11667	130	11	.	.	.	.	.	.	.	.	.	.	.	.	.

2·785 = sum of the tangents. ·011

·011 = sum of the differences.

2·774 = sum of the sines.

16. For the sum of the sines below P in the S.E. quarter.

				1	2	3	4	5	6	7	8	9	10	11	12	
7	4333	80	·018	.	.	.	.	.	.	.	.	.	.	.	.	.
8	5000	290	58	.	.	.	.	.	.	.	.	.	.	.	.	.
9	5667	1810	319	.	.	.	1	.	.	.	.	.	.	.	.	·001
10	6333	3280	518	.	.	1	.	.	.	1	.	.	.	.	.	2
11	7000	4640	663	.	1	.	.	.	1	.	.	.	.	.	1	3
12	7667	5590	729	.	.	1	.	.	1	.	.	.	.	.	1	3
13	8333	5830	700	.	.	.	1	.	.	1	.	.	.	.	1	3
14	9000	5700	633	.	.	.	1	.	.	.	1	.	.	1	.	3
15	9667	5240	542	.	.	.	.	.	.	1	.	.	1	.	1	3
16	10333	4560	441	.	.	.	.	.	.	.	1	.	.	1	.	2
17	11000	3920	356	.	.	.	.	.	.	.	.	1	.	.	1	2
18	11667	3180	273	.	.	.	.	.	.	.	.	.	.	1	.	1
19	12333	2630	213	.	.	.	.	.	.	.	.	.	.	1	.	1
20	13000	2230	172	.	.	.	.	.	.	.	.	.	.	1	.	1

5·635 = sum of the tangents. ·025

·025 = sum of the differences.

5·610 = sum of the sines.

Having now obtained the sums of the sines for the several quadrants, the next business is to collect them together, and deduct the negatives from the affirmatives. Now this may be done either for each observatory separately, or for both together. It is here done separately, in order from thence to discover also the ratio of their effects.

And first for the Southern observatory o.

Affirmatives.			Negatives.		
1 . . 24·795	N.W.	Alt.	3 . . 2·374	S.W.	Alt.
2 . . 19·792	N.E.		4 . . 0·375	S.E.	
7 . . 24·806	S.W.	Dep.	5 . . 13·534	N.W.	Dep.
8 . . 29·213	S.E.		6 . . 12·356	N.E.	
98·606 = sum of affirm.			28·639 sum.		

28·639 = sum of negat.

69·967 = effective sum of the sines for o.

Secondly, for the Northern observatory P.

Affirmatives.			Negatives.		
11 . . 25·078	S.W.	Alt.	9 . . 0·019	N.W.	Alt.
12 . . 20·261	S.E.		10 . . 0·090	N.E.	
13 . . 25·637	N.W.	Dep.	15 . . 2·774	S.W.	Dep.
14 . . 26·161	N.E.		16 . . 5·610	S.E.	
97·137 = sum of affirm.			8·493 sum.		

8·493 = sum of negat.

88·644 = effective sum of the sines for P.

69·967 = the same for o.

158·611 = the sum of the sines for both observ.

From these numbers it appears, that the effect of the attraction at the Northern observatory is to that at the Southern one, nearly as 70 is to 89, or as 7 to 9 nearly. This difference is to be attributed chiefly to the effect of the hills on the South of the Southern observatory, which were considerably larger and nearer to it, than those on the back of the Northern observatory. For though the Southern observatory was placed 273 feet above the level of the Northern one, which removed it considerably more above the centre of gravity of the hill than the latter, it was at the same time placed consi-

derably nearer than the other to the middle in a horizontal direction; so that probably the one difference nearly balances the other; and accordingly we find that the sum of the affirmative altitudes for o is 44·587, and of those for P 45·339, which differ by only a 60th part nearly.

It only remains now to multiply the sum of the sines by the common breadth of the rings, and by the common difference of the sines of the angles made by the meridian and the several radii. It has already been observed, that the former is $666\frac{2}{3}$, and the latter $\frac{1}{12}$; therefore $\frac{1}{12} \times 666\frac{2}{3} = \frac{2000}{36} = \frac{500}{9}$ is their product: consequently, $158 \cdot 611 \times \frac{500}{9} = 8811\frac{2}{3}$ nearly, is the sum of the two opposite attractions, made by the hill &c, at the two observatories.

In order now to compare this attraction with that of the whole earth; this body may be considered as a sphere, and the observatories as placed at its surface; since the very small differences of these suppositions from the truth, are of no consequence at all in this comparison. Now the attraction of a sphere, on a body at its surface, is known to be $= \frac{2}{3}cd$, where d is $=$ the diameter of the sphere, and $c = 3 \cdot 1416 =$ the circumference of the circle to the diameter 1. But cd is $=$ the circumference of the circle to the diameter d; and therefore the attraction of a sphere will be expressed by barely $\frac{2}{3}$ of its circumference; which is a theorem well adapted to the present computation. The length of a degree in the mean latitude of 45°, is 57028 French toises (see p. 327, Phil. Trans. 1768): and the same result nearly is obtained by taking a mean among all the measures of degrees there set down, that mean being 57038 toises. I shall therefore use the round number 57030 as probably nearer the truth. This number being multiplied by 6, the product 342180 shows the number of French feet in one degree; but, by p. 326 of the same volume, the lengths of the Paris and London feet are as 76·734 to 72, that is, as 4·263 to 4; therefore, as 4 : 4·263 : : 342180 : 364678 = the English feet in one degree; and this being multiplied by 360 the whole number of degrees, there results 131284080 feet for the whole circumference, which are equal

to $24864\frac{1}{2}$ miles, making $69\frac{1}{15}$ to a degree in the mean latitude. Lastly, $\frac{2}{3}$ of 131284080 give 87522720 for the measure of the attraction of the whole earth.

Consequently, the whole attraction of the earth, is to the sum of the two contrary attractions of the hill, as the number 87522720 to $8811\frac{2}{3}$, that is, as 9933 to 1 very nearly, on supposition that the density of the matter in the hill, is equal to the mean density of that in the whole earth.

But the Astronomer Royal found, by his observations, that the sum of the deviations of the plumb line, produced by the two contrary attractions, was 11·6 seconds. Hence then it is to be inferred, that the attraction of the earth, is actually to the sum of the attractions of the hill, nearly as radius to the tangent of 11·6 seconds, that is, as 1 to ·000056239, or as 17781 to 1; or as 17804 to 1 nearly, after allowing for the centrifugal force arising from the rotation of the earth about its axis.

Having now obtained the two results, namely, that which arises from the actual observations, and that belonging to the computation on the supposition of an equal density in the two bodies, the two proportions compared must give the ratio of their densities, which accordingly is that of 17804 to 9933, or 1434 to 800 nearly, or almost as 9 to 5. And so much does the mean density of the earth exceed that of the hill.

Thus then we have at length obtained the object which we have been in quest of, through the very laborious calculations that have been described in this tract, and in the survey and measurements from which these computations were made; namely, the ratio of the mean density of all the matter in the earth, in comparison with the density of the matter of which the hill is composed; which ratio we have found to be equal to that of 9 to 5. And, for the reasons before mentioned, I think we may rest satisfied, that this proportion is obtained to a considerable degree of proximity, probably to within the 50th part, if not the 100th part of its true magnitude.

Another question however still arises, namely, what is the density of the matter in the hill? Is its mean density equal

to that of water, of sand, of clay, of chalk, of stone, or of some of the metals? For, according to the matter, or different sorts of matter, of which it is formed, and according as it is constituted with or without large vacuities, its mean density may be greater or less, and that in a degree which is not certainly known. A considerable degree of accuracy in this point could perhaps only be obtained by a close examination of the internal structure of the hill. And the easiest method of doing this would be to procure holes to be bored, in several parts of it, from the surface to a sufficient depth, after the manner that is practised in boring holes to the coal mines from the surface of the ground; for by such operation it is known what kind of strata the borer has passed through, together with their dimensions and densities. The proper mean among all these would be the mean density of the hill, as compared to water, or to any other simple matter; and thence we should obtain the comparative density of the whole earth with respect to water. But, in the present instance, we must be satisfied with the estimate arising from the report of the external view of the hill; which is, that to all appearance it consists of an intire mass of solid rock.

The following is an extract of a letter from a learned gentleman, Mr. Duncan Macara of Fortingal, who wrote the account of the parish in which the mountain is situated, for the use of Sir John Sinclair's Statistical Work. "As to the mountain, says Mr. Macara, I have been at the top of it, and round the bottom frequently, but have not observed any thing particular on it, that was not to be found in the mountains all around, some of which are higher than it. I have seen gentlemen, who made natural history their study, who were of the same opinion, and they examined it narrowly. On the north side it is covered with moss, generally about two or three inches, on which grows heath, and in some parts a little grass. On the south and west are cairns, of large and small stones, up to the summit. How they came to be so high, is a question, if an earthquake was not the cause. The stone, of which the mountain seems to be one solid mass, is the same

in appearance as all the other Grampian hills, a black solid stone, below the surface. There are no volcanic appearances here, or in any of the mountains of this county, so far as I know, or have been told. At the bottom of Shichallin, to a great distance, there are lime rocks, jutting out here and there. As to plants, there are many of the Alpine kind, which grow there, and on the other moutains thereabouts, &c." Whence we seem authorized to infer, that the mountain consists chiefly of granitic black rock, the specific gravity of which approaches nearly to 3, that is 3 times the density of water.

Again, Mr. Playfair, the learned professor of natural philosophy in the university of Edinburgh, has since made a mineralogical survey of the hill, by which he has discovered, that the varieties of rock of which it consists, may be reduced to three kinds: a granular quartz, which occupies all the middle part of the mountain; a micaceous schistus, which encompasses the former nearly all round like a zone to within 600 feet of the bottom; and lastly a calcareous zone, which may be said to surround the mountain at its base. Though there is some irregularity in the disposition of these zones, this is at least, a general idea of the structure of the mountain, which does not differ greatly from the truth. By what Mr. Playfair could conjecture, the mean specific gravity of the whole would be about 2·7 or 2·8, one stratum being about 2·4, another about 2·75, and some rocks as high as 3, and even 3·2. On the whole then it appears not unreasonable to suppose the mean specific gravity of the mountain to be from 2·7 to 2·75 or $2\frac{3}{4}$. Now $\frac{9}{5} \times 2\frac{3}{4}$ gives $\frac{99}{20}$ or almost 5; that is, under these circumstances, the medium density or specific gravity of the whole mass of the earth, in proportion to that of water, is nearly as 5 to 1, or that it is about 5 times the weight of water. And, with regard to this particular experiment, it appears that we may rest satisfied with this general result, drawn from a medium specific gravity among all its constituent rocks. For, to have instituted a calculation founded on an assumed distribution of the hill, in several zones of different densities, I am of opinion would have been

a mere useless labour, as incompatible with the unknown quantities and positions of the various kinds of the rocks, and also with respect to the degree of accuracy to be counted on, in the observations and measurements made by the first conductors of this important experiment.

To what useful purposes the knowledge of the mean density of the earth, as above found, may be applied, it is not necessary here to show. I shall therefore conclude this tract with a reflection or two on the premises that have been delivered. Sir Isaac Newton thought it probable, that the mean density of the earth might be 5 or 6 times as great as the density of water; and we have now found, by experiment, that it is very little less than what he had thought it to be: so much justness was there even in the surmises of this wonderful man! Since then the mean density of the whole earth is about double that of the general matter near the surface, and within our reach, it follows, that there must be somewhere within the earth, toward the more central parts, great quantities of metals, or such like dense matter, to counterbalance the lighter materials, and produce such a considerable mean density on the whole. If we suppose, for instance, the density of metal to be 10, which is about a mean among the various kinds of it, the density of water being 1, it would require 16 parts out of 27, or considerably more than one-half of the matter in the whole earth, to be metal of this density, in order to compose a mass of such mean density as we have found the earth to possess by the experiment: or $\frac{4}{13}$, or between $\frac{1}{3}$ and $\frac{1}{4}$ of the whole magnitude will be metal; and consequently $\frac{20}{31}$, or nearly $\frac{2}{3}$ of the diameter of the earth, is the central or metalline part. But if the metalline matter be chiefly iron, which as far as we know is by much the most predominant metal, then the half of the whole terrestrial magnitude would be the bulk of the ferruginous matter.

Another inference that readily occurs, is this: viz, that thus knowing the mean density of the earth in comparison with water, and the densities of all the planets relatively to the earth, we can now assign the proportions of the densities

of them all, as compared to water, after the manner of a common table of specific gravities. And the numbers expressing their relative densities, in respect of water, will be as here annexed, supposing the densities of the planets, as compared to each other, to be as laid down in Mr. De la Lande's astronomy.

Water . . .	1
The Sun . .	$1\frac{4}{15}$
Mercury . .	$10\frac{1}{6}$
Venus . . .	$6\frac{1}{3}$
The Earth . .	5
Mars	$3\frac{2}{7}$
The Moon . .	$3\frac{5}{11}$
Jupiter . . .	$1\frac{1}{6}$
Saturn . . .	$\frac{5}{11}$

Thus then we have brought to a conclusion the computation of this important experiment, and, it is hoped, with no inconsiderable degree of accuracy.—But this is the first experiment of the kind, which has been so minutely and circumstantially treated; and first attempts are seldom so perfect and just, as succeeding endeavours afterwards render them. And besides, a frequent repetition of the same experiment, and a coincidence of results, afford that firm dependance on the conclusions, and satisfaction to the mind, which can scarcely ever be obtained from a single trial, however carefully it may be executed. For these reasons it is to be wished, that the world may not rest satisfied barely with what has been done in this instance, but that they will repeat the experiment in other situations, and in other countries, with all the care and precision that it may be possible to give it; till a uniformity of conclusions shall be found, sufficient to establish the point in question, beyond any reasonable probability of doubt. What has been already done in the present case, will render any future repetition more easy and perfect. But improvements may doubtless be made, perhaps both in the mode of computation, and in the survey; in the latter especially there certainly may. Some improvements of this kind

have been hinted at in different parts of this tract, which, with others, are here collected together, that they may readily be seen in one point of view. They are principally these. Procure one base, or more if convenient, very accurately measured, in such situation, that as many more points as possible in the survey may be seen from it. Assume as many principal or eminent points and objects as may be proper and convenient; and from each one of them, measure the angles formed by all the rest that can be seen, both horizontal and vertical angles; and repeat these observations, if convenient, with the instrument varied or reversed, taking the means among the several quantities of each angle. Take then as many sections of the ground, and as far extended in all directions, as the time and circumstances will possibly admit. Of the sections, those that are horizontal, or level, are the best, as they require no calculation; procure therefore as many as possible of them. In vertical sections, observe the vertical angles, not in the plane of the section, but at some other point of which the bearing is also taken from the beginning of the section line, and where the horizontal angles of the poles are taken, for the reasons before mentioned in p. 22. And it will be a still further convenience if the section be made in such direction, as to form a right angle with the line drawn to the point, or station, from which the vertical angles of the poles are observed, as may be seen from what is said in p. 21. It might perhaps be proper to make some experiments on a valley, instead of a hill, taking two observatories at the two opposite sides of it, both for the greater variety in this interesting problem, and because also the survey would be more easily made, on account of the ground being more in view at each station, than in the case of a hill, which generally hides more than half the compass from the observer. In computing the relative altitudes of all the principal stations, let the operations be performed mutually both backward and forward, that is, from both of every two objects, having for that purpose observed at each of them the vertical angle of the other, namely, both the

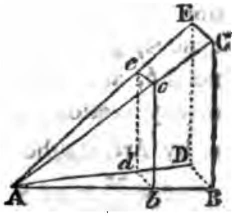
angle of elevation and the angle of depression, and take the mean between the two computed differences of altitude; for this excludes the necessity of making the proper allowances for refraction, and for the curvature of the earth; since the effect of each of these is balanced and corrected by that of the counter observation. But as to those points in the sections which are far distant from the observer, and where great accuracy is required, it may be proper to make the allowance for refraction and curvature, as there is generally no back observation, by which their effects may be balanced. These are the chief hints which at present occur, besides the general information to be derived by the computer from the perusal of the modes of computation that have been described in this tract. As to the surveyor, he will strike out other convenient ways of measurement, adapted to the circumstances with which the nature of the survey may happen to be attended.

A map of the country about Shichallin is hereunto annexed, (pl. 4), to convey a general idea of the nature of the ground, and for the better illustration of the description given in the former parts of this tract.

TRACT XXVII.

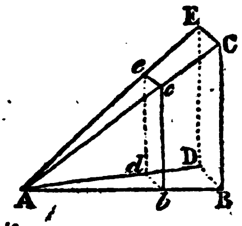
CALCULATIONS TO DETERMINE AT WHAT POINT IN THE SIDE OF A HILL ITS ATTRACTION WILL BE THE GREATEST, &c. READ AT THE ROYAL SOCIETY, NOV. 11, 1779.

Art. 1. The great success of the experiment, lately made by the Royal Society, on the hill Shihallin, to determine the universal attraction of matter, and the important consequences that have resulted from it, may probably give occasion to other experiments of the same kind to be made elsewhere: and as all possible means of accuracy and facility are to be desired, in so delicate and laborious an undertaking; it has occurred that it might not be unuseful to add, by way of supplement to my paper of calculations relative to the above-mentioned experiment, an investigation of the height above the bottom of a hill, at which its horizontal attraction shall be the greatest; since that is the height at which commonly the observations ought to be made, and since this best point of observation has never been any where determined, that I know of, but has been variously spoken of or guessed at, being sometimes accounted at $\frac{1}{3}$, and sometimes at $\frac{1}{2}$ of the height of the hill; whereas, from these investigations, it is found to be generally at about only $\frac{1}{4}$ of the altitude from the bottom.



2. Let ABCEDA be part of a cuneus of matter, its sides or faces being the two similar right-angled triangles ABC, ADE, meeting in the point A, and forming the indefinitely small angle BAD. Then, of any section $bced$, perpendicular to the planes ABD, ADE, the attraction on a body at A, in the direction

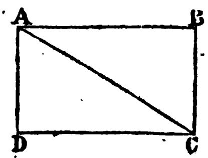
AB, is equal to the constant quantity ss; where s denotes the sin. ∠BAC, and s the sin. ∠BAD, to the radius 1.



For, first, since the magnitude of the flowing section is every where as Ab^2, and the attraction of the particles of matter inversely as the same, or as $\frac{1}{Ab^2}$; therefore their product or $\frac{Ab^2}{Ab^2}$ or 1, a constant quantity, is as the force of attraction of *bced*.

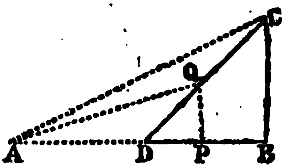
Now, to find what that quantity is. Put $AB = a$, and $BC = x$; then BD or CE, the distance between the two planes at the distance AB is $= as$. But the force of a particle in the line CE is as $\frac{1}{AC^2}$ in the direction AC, and therefore it is as $\frac{AB}{AC^3}$ in the direction AB; consequently the force of the whole lineola CE, in the direction AB, is $\frac{AB \cdot CE}{AC^3}$; and therefore the fluxion of the force of the section BCED, or $\dot{f}$, is $= \frac{AB \cdot CE}{AC^3} \cdot \dot{BC} = \frac{a \cdot as \cdot \dot{x}}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{a^2 s\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}}$; and the fluent gives $f = \frac{sx}{\sqrt{(a^2 + x^2)}} = s \times \frac{BC}{AC} = ss$ for the attraction itself.

3. To find now the attraction of the whole right-angled cuneus, on a body at A, in the direction AB.—Since the force of each section is ss by the last article; therefore the force of all the sections, the number of them being AB or a, is $ass = s \cdot AB \cdot \frac{BC}{AC}$, the force of the whole cuneus ABCEDA.

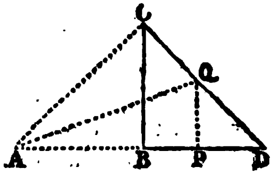


4. To find the attraction of the rectangular part ABCD, on A, in the direction AB; ABCD being one side of the cuneus, and AD its edge.—Put $AD = BC = b$, and $AB = x$. Then, the force of any section, as BC, being always as ss by Art. 2, the fluxion of the force, or $\dot{f}$, will be $= s\dot{x} = s\dot{x} \times \frac{BC}{AC} = s\dot{x} \times \frac{b}{\sqrt{(b^2 + x^2)}} = \frac{bs\dot{x}}{\sqrt{(b^2 + x^2)}}$; and the fluent is

$f = bs \times$ hyp. log. of $\frac{x+\sqrt{(b^2+x^2)}}{b} = s \cdot \text{BC} \times$ hyp. log. $\frac{\text{AB}+\text{AC}}{\text{BC}}$, = the attraction of ABCD.



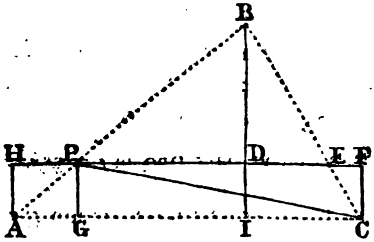
5. To find the attraction of the right-angled part BCD, of a cuneus whose edge passes through A, the place of the body attracted.—Put $\text{AB}=a$, $\text{BC}=b$, $\text{BD}=c$, $\text{DA}=d=a-c$, $\text{DC}=e$, $\text{AC}=g$, and $\text{DP}=x$. Then, the force of any section PQ being still as ss, the fluxion of the force of the part DPQ is $\dot{f} = ss\dot{x} = s\dot{x} \times \frac{\text{PQ}}{\text{AQ}} = \frac{bsx\dot{x}}{\sqrt{(c^2d^2+2c^2dx+e^2x^2)}}$, and the correct fluent, when $x=c$, is $f = \frac{bcs}{ee} \times (g - d - \frac{dc}{e} \times$ hyp. log. $\frac{ee+eg+dc}{de+dc}) =$ the force of a body at A in the direction AB.



6. To find the attraction of the right-angled part BCD, on the point A.—Using here again the same notation as in the last article, we have $\dot{f} = ss\dot{x} = s\dot{x} \times \frac{\text{PQ}}{\text{AQ}} = \frac{bsx\dot{x}}{\sqrt{(c^2d^2-2c^2dx+e^2x^2)}}$. The correct fluent of which, when $x=c$, is $f = \frac{bcs}{ee} \times (g - d + \frac{dc}{e} \times$ hyp. log. $\frac{ee+eg-dc}{de-dc})$.

7. To apply now these premises, to the finding of the place where the attraction of a hill is greatest, it will be necessary to suppose the hill to have some certain figure. That position is most convenient for observing the attraction, in which the hill is most extended in the east and west direction. Supposing then such a position of a hill, and that it is also of a uniform height and meridional section throughout; the point of observation must evidently be equally distant from the two ends. But instead of being only considerably extended, I shall suppose the hill to be indefinitely extended to the east, and to the west of the point of observation, in order that the investigation may be nearly

mathematically true, and yet at the same time sufficiently exact for the before-said limited extent also. It will also come nearest to the practical experiment, to suppose the hill to be a long triangular prism, so that all its meridional sections may be similar triangles. Let therefore the triangle ABC represent its section, by a vertical plane passing through the meridian, or one side of an indefinitely thin cuneus, whose edge is in BG; or rather PBCGP the side of one cuneus, and PAG the side of another, their common edge being the line PG, perpendicular to the base AC; P being the required point in the side AB, where the attraction of the section ABC, or indefinitely thin cuneus, shall be greatest, in a direction parallel to the horizon AC. And then, from the foregoing suppositions, it is evident that, in whatever point of AB the attraction of ABC is greatest, there also will the attraction of the whole hill be also the greatest, very nearly.



8. Now draw HPDEF parallel to the base AC; and AH, PG, BI, CF, perpendicular to the same. Then it is evident that at the point P, in the direction PF, the attraction of PBCGP is affirmative, and that of PAG negative. But PBCGP is = PBD + BDE + PFCG − EFC; and PAG = PHAG − PHA. Therefore the attractions of PBD, BDE, PFCG, PHA, are affirmative; and those of EFC, PHAG, negative.

Put now $BI = a$, $AI = b$, $IC = c$, $AB = d$, $BC = e$, $AC = g = b + c$, and $PG = x$, the altitude of the point P above the bottom. Also let s = the sine of the indefinitely small angle of the cuneus, to radius 1; and $q = \sqrt{(d^2g^2 - 2abgx + d^2x^2)}$,

Then by Art. 3, the attraction

$$\text{of} \left\{ \begin{array}{l} PBD \text{ is } s \cdot PD \cdot \frac{BD}{BP} = sb \times \frac{a-x}{d}, \\ PHA \text{ is } s \cdot PH \cdot \frac{PG}{PA} = sb \times \frac{x}{d}. \end{array} \right\}$$

By Art. 4, the attraction

$$\text{of}\left\{\begin{array}{l}\text{PFCQ is } s\,.\,\text{PG}\times\text{h. l. }\dfrac{\text{PF}+\text{PC}}{\text{PG}}=sx\times\text{h. l. }\dfrac{ag-bx+qq}{ax},\\ \text{PGAH is } s\,.\,\text{PG}\times\text{h. l. }\dfrac{\text{PH}+\text{PA}}{\text{PG}}=sx\times\text{h. l. }\dfrac{b+d}{a}.\end{array}\right\}$$

By Art. 5, the attraction of EFC is

$$s\,.\,\frac{\text{EF}\,.\,\text{FC}}{\text{EC}^2}\times(\text{PC}-\text{PE}-\frac{\text{PE}\,.\,\text{EF}}{\text{EC}}\times\text{h. l. }\frac{\text{EC}^2+\text{EC}\,.\,\text{PC}+\text{PE}\,.\,\text{EF}}{\text{PE}\,.\,\text{EC}+\text{PE}\,.\,\text{EF}}=\frac{sc}{ee}\times$$

$$[qq-g(a-x)-gc\,.\,\frac{a-x}{e}\times\text{h. l. }\frac{acg+eqq+acx-bcx}{g\,.\,(c+e)\,.\,(a-x)}].$$

Lastly, by Art. 6, the attraction of BDE is

$$s\,.\,\frac{\text{BD}\,.\,\text{DE}}{\text{BE}^2}\times(\text{PB}-\text{PE}+\frac{\text{PE}\,.\,\text{DE}}{\text{BE}}\times\text{h. l. }\frac{\text{BE}^2+\text{BE}\,.\,\text{BP}-\text{PE}\,.\,\text{DE}}{\text{PE}\,.\,\text{BE}-\text{PE}\,.\,\text{DE}})=\frac{sc}{ee}\,.$$

$$(a-x)\times(d-g+\frac{eg}{e}\times\text{h. l. }\frac{ee+de-cg}{eg-cg}).$$

These quantities being collected together, with their proper signs, and contracted, we have

$$s\times\left\{\begin{array}{l}\dfrac{ab}{d}+c\,.\,\dfrac{ad-qq-dx}{ee}+x\times\text{hyp. log. }\dfrac{ag+qq-bx}{(b+d)\,x}+\\ \dfrac{ccg\,.\,(a-x)}{e^3}\times\text{h. l. }\dfrac{(ce+de-cg)\,.\,(acg+eqq+acx-bcx)}{gg\,.\,(ee-cc)\,.\,(a-x)}\end{array}\right\},$$

for the whole attraction in the direction PE.

9. Having now obtained a general formula for the measure of the attraction, in any sort of triangle, if the particular values of the letters be substituted, which any practical case may require, and the fluxion of this attraction be put $= 0$, the root of the resulting equation will be the required height, from the bottom of the hill.

10. But for a more particular solution in simpler terms, let us suppose the triangle ABC to be isosceles; in which case we shall have $d=e$, and $g=2b=2c$, and then the above general formula will become

$$s\times\left\{\begin{array}{l}\dfrac{2ad-qq-dx}{dd}\,b+x\times\text{h. l. }\dfrac{2ab+qq-bx}{(b+d)\,.\,x}\\ +\dfrac{a-x}{d^3}\,.\,2b^3\times\text{h. l. }\dfrac{2ab^2+dq^2-(b^2-a^2)\,.\,x}{2b^2\,.\,(a-x)}\end{array}\right\},$$

for the value of the attraction in the case of the isosceles triangle, where q^2 is $=\sqrt{(4a^2b^2-4ab^2x+d^2x^2)}$. And the fluxion of this expression being equated to 0, the equation

will give the relation between a and x, for any values of b and d, by a process not very troublesome.

11. Now it is probable that the relation between a and x, when the attraction is greatest, will vary with the various relations between b and d, or between b and a. Let us therefore find the limits of that relation, between which it may always be taken, by using two particular extreme cases, the one in which the hill is very steep, and the other in which it is very flat, or a very small in respect of b or d.

12. And first let us suppose the triangular section to be equilateral; in which case the angle of elevation is 60°, which being a degree of steepness that can scarcely ever happen, this may be accounted the first extreme case. Here then we shall have $d = 2b = \frac{2}{3}a\sqrt{3}$, and the formula in Art. 10, will become $s \times (\frac{2a-r-x}{2} + x \times \text{h. l.} \frac{2a+2r-x}{3x} + \frac{a-x}{4} \times \text{h. l.} \frac{a+2r+x}{a-x})$, for the value of the attraction in the case of the equilateral triangle, in which r is $= \sqrt{(a^2 - ax + x^2)}$.

13. Or if we take $x = na$, where n expresses what part of a is denoted by x, the last formula will become $sa \times [1 - \frac{1}{2}n - \frac{1}{2}\sqrt{(1-n+n^2)} + n \times \text{h. l.} \frac{2-n+2\sqrt{(1-n+n^2)}}{3n} + \frac{1-n}{4} \times \text{h. l.} \frac{1+n+2\sqrt{(1-n+n^2)}}{1-n}]$, for the case of the equilateral triangle.

14. To find the maximum of the expression in the last article, put its fluxion $= 0$, and there will result this equation, $1 + \frac{1+n}{\sqrt{(1-n+n^2)}} = 2\,\text{hyp. log.} \frac{2-n+2\sqrt{(1-n+n^2)}}{3n} - \frac{1}{2}\,\text{h. l.} \frac{1+n+2\sqrt{(1-n+n^2)}}{1-n}$; the root of which is $n = .251999$. Which shows that, in the equilateral triangle, the height from the bottom, to the point of greatest attraction, is only $\frac{1}{500}$th part more than $\frac{1}{4}$ of the whole altitude of the triangle. And this is the limit for the steepest kind of hills.

15. Let us find now the particular values of the measure of attraction, arising by taking certain values of n, varying

by some small difference, in order to discover what part of the greatest attraction is wanting by observing at different altitudes.

16. And first using the value of n, (·251999) as found in the 14th article, the general formula in Art. 13, gives $sa \times 1·0763700$, for the measure of the greatest attraction.

17. If $n = \frac{3}{10}$, or $x = \frac{3}{10}a$; the same formula gives $\frac{sa}{20} \times (17 - \sqrt{79} + 6\text{h. l.} \frac{17 + 2\sqrt{79}}{9} + \frac{1}{2}\text{hyp.log.} \frac{13 + 2\sqrt{79}}{7}) = sa \times 1·0702512$ for the attraction at $\frac{3}{10}$ of the altitude, which is something less than the former.

18. If $n = \frac{4}{10} = \frac{2}{5}$; the formula gives $\frac{sa}{20} \times (16 - \sqrt{76} + 8\text{h. l.} \frac{8 + \sqrt{76}}{6} + 3\text{h. l.} \frac{7 + \sqrt{76}}{3}) = sa \times 1·0224232$, for the attraction at $\frac{4}{10}$ or $\frac{2}{5}$ of the altitude; less again than the last was.

19. If $n = \frac{5}{10} = \frac{1}{2}$; the formula gives $\frac{1}{4}sa \times [3 - \sqrt{3} - 2\text{h. l.} 3 + \frac{1}{2}\text{h. l.}(3 + 2\sqrt{3})] = sa \times ·9340963$ for the attraction at half way up the hill; still less again than the last.

20. If $n = \frac{6}{10} = \frac{3}{5}$; the formula gives $\frac{sa}{20} \times (14 - \sqrt{76} + 12\text{ h. l.} \frac{7 + \sqrt{76}}{9} + 2\text{ h. l.} \frac{8 + \sqrt{76}}{2}) = sa \times ·8109843$, for the attraction at $\frac{6}{10}$ or $\frac{3}{5}$ of the altitude from the bottom; being still less than the last was. And thus the quantity of attraction is continually less and less, the higher we ascend up the hill, above the ·251999 part, or in round numbers ·252 part of the altitude. Let us now descend, by trying the numbers below ·252; and first,

21. If $n = ·25 = \frac{1}{4}$; the same formula in Art. 13, gives $\frac{1}{8}sa \times (7 - \sqrt{13} + 2\text{ h. l.} \frac{7 + 2\sqrt{13}}{3} + \frac{1}{4}\text{ h. l.} \frac{5 + 2\sqrt{13}}{3}) = sa \times 1·0763589$, for the attraction at $\frac{1}{4}$ of the altitude; and is very little less than the maximum.

22. If $n = \frac{2}{10} = \frac{1}{5}$; the formula gives $\frac{1}{10}sa \times (9 - \sqrt{21} + 2\text{ h. l.} \frac{9 + 2\sqrt{21}}{3} + 2\text{ h. l.} \frac{3 + \sqrt{21}}{2}) = \frac{1}{10}sa \times (9 - \sqrt{21} + 2\text{ h. l.} \frac{23 + 5\sqrt{21}}{2}) = sa \times 1·0684622$, for the attraction at $\frac{2}{10}$ or

$\frac{1}{5}$ of the altitude; and is something less than at $\frac{1}{4}$ of the altitude.

23. If $n = \frac{1}{10}$; the formula gives $\frac{sa}{20} \times (19 - \sqrt{91} + 3\,\text{h. l.}\,\frac{19+2\sqrt{91}}{3} + \frac{9}{2}\,\text{h. l.}\,\frac{11+2\sqrt{91}}{9}) = sa \times \cdot 9986188$; for the attraction at $\frac{1}{10}$ of the altitude; still less than the last was, And, lastly,

24. If $n = 0$, or the point be at the bottom of the hill; the formula gives $\frac{1}{4}sa \times (2 + \text{h. l. } 3) = sa \times \cdot 7746531$, for the attraction at the bottom of the hill; which is between $\frac{2}{7}$ and $\frac{1}{4}$ of the greatest attraction, being something greater than $\frac{2}{7}$ but less than $\frac{1}{4}$ of it.

25. The annexed table exhibits a summary of the calculations made in the preceding articles; where the first column shows at what part of the altitude of the hill the observation is made; the second column contains the corresponding numbers which are proportional to the attraction; and the third column shows what part of the greatest attraction is lost at each respective place of observation, or how much each is less than the greatest.

$\frac{6}{10}$	8109843	$\frac{1}{4}$
$\frac{5}{10}$	9340963	$\frac{2}{15}$
$\frac{4}{10}$	10224232	$\frac{1}{20}$
$\frac{3}{10}$	10702512	$\frac{1}{180}$
$\frac{252}{1000}$	10763700	0
$\frac{1}{4}$	10763589	$\frac{1}{97852}$
$\frac{2}{10}$	10684622	$\frac{1}{134}$
$\frac{1}{10}$	9986188	$\frac{1}{14}$
0	7746531	$\frac{2}{7}$

26. Having now so fully illustrated the case of the first extreme, or limit, let us search what is the limit for the other extreme, that is, when the hill is very low or flat. In this case b is nearly equal to d, and they are both very great in respect of a; consequently the formula for the attraction in Art. 10, will become barely $s \times [x \times \text{h. l.}\,\frac{2a-x}{x} + 2(a-x) \times \text{h. l.}\,\frac{2a-x}{a-x}]$; the fluxion of which being put $= 0$, we obtain $0 = \text{h. l.}\,\frac{2a-x}{x} - 2\,\text{h. l.}\,\frac{2a-x}{a-x} = \text{h. l.}\,\frac{2a-x}{x} - \text{h. l.}\,(\frac{2a-x}{a-x})^2 = \text{h. l.}\,\frac{(a-x)^2}{x(2a-x)}$; hence therefore $(a-x)^2 = x\,(2a-x)$, and

$x = a \times (1 - \sqrt{\frac{1}{2}}) = \cdot 2929a$. Which shows that the other limit is $\frac{29}{100}$; that is, when the hill is extremely low, the point of greatest attraction is at $\frac{29}{100}$ of the altitude, like as it is at $\frac{25}{100}$ when the hill is very steep. And between these limits it is always found, it being nearer to the one or the other of them, as the hill is flatter or steeper.

27. Thus then we find that at $\frac{1}{4}$ of the altitude, or very little more, is the best place for observation, to have the greatest attraction, from a hill in the form of a triangular prism, of an indefinite length. But when its length is limited, the point of greatest attraction will descend a little lower; and the shorter the hill is, the lower will that point descend. For the same reason, all pyramidal hills have their place of greatest attraction a little below that above determined. But if the hill have a considerable space flat at the top, after the manner of a frustum, then the said point will be a little higher than as above found. Commonly, however, $\frac{1}{4}$ of the altitude may be used for the best place of observation, as the point of greatest attraction will seldom differ sensibly from that place. And when uncommon circumstances may produce a difference too great to be intirely neglected, the observer must exercise his judgment in guessing at the necessary change he ought to make, in the place of observation, so as to obtain the best effect which the concomitant circumstances will admit of.

TRACT XXVIII.

OF CUBIC EQUATIONS AND INFINITE SERIES. READ AT A MEETING OF THE ROYAL SOCIETY, JUNE 1, 1780.

The following pages are not to be understood as intended to contain a complete treatise on cubic equations, with all the methods of solution that have been delivered by other writers; but they are chiefly employed on the improvements of some properties that were before but partially known, with the discovery of several others, which appear to be new, and of no small importance: for I have only slightly mentioned such of the generally known properties as were necessary to the introduction, or investigation, of the many curious consequences herein deduced from them.

Art. 1. Every equation, whose terms are expressed in simple integral powers, has as many roots as there are dimensions in the highest power. And when all the terms are brought to one side of the equation, and the coefficient of the first term, or highest power, is $+1$, then the coefficient of the second term, is equal to the sum of all the roots with contrary signs: the coefficient of the third term, is equal to the sum of all the products made by multiplying every two of the roots together; the coefficient of the fourth term, equal to the sum of all the products arising from the multiplication of every three of the roots together, &c; and the last term, equal to the continual product of all the roots; the signs of all of them being supposed to be changed, into the contrary signs, before these multiplications are made. All this is evident from the generation of equations. And, from these properties of the coefficients, the following deductions are easily made.

2. If the signs of all the roots of an equation be changed, and another equation be generated from the same roots, with the signs so changed; the terms of this last equation will have the same coefficients as the former; only, the signs of all the even terms will be changed, but not those of the odd terms: for the coefficients of the second, fourth, and the other even terms, are made up of products consisting each of an odd number of factors; while those of the third, fifth, and other odd terms are composed of products having an even number of factors: and the change of the signs of all the factors produces a change in the sign of the continual product of an odd number of factors, but no change in the sign of that of an even number of factors. Therefore, changing the signs of all the even terms, namely, the second, fourth, &c, produces no alteration in the roots, but only in their signs, the positive roots being changed into negative, and the negative into positive. But by changing any or all the signs of the odd terms, the equation will no longer have the same roots as before, but will have new roots of very different magnitudes from those of the former, unless the sign of the first term, or highest power, be changed also; but this term is always to be supposed to remain positive.

3. It also follows, that when any term is wanting in an equation, or the coefficient of any term is equal to 0, the sum of the negative products, in the coefficient of that term, is equal to the sum of the positive products in the same. And if it be the second term which is wanting, then the equation has both negative and positive roots, and the sum of the negative roots is equal to the sum of the positive ones. But if it be the last term which is wanting, then one of the roots of the equation is equal to nothing. And hence arises a method of transforming any equation into another which shall want the second term: and to this latter state it will be proper to transform every cubic equation, before we attempt the solution of it.

4. Let therefore $x^3 + px = q$ be such a cubic equation,

wanting the second term, where p and q represent any numbers, positive or negative.

5. Now from the premises it follows, that this equation has three roots; that some are positive, and others negative; that two of them are of one affection, and are together equal to the third of a contrary affection, namely, either two negative roots, which are together equal to the other positive, or two positive roots equal to the third negative.

6. But the signs of the three roots are easily known from the sign of the quantity q; the sign of the greatest root being the same with the sign of q, when this quantity is on the right-hand side of the equation, and the other two roots of the contrary sign. For when q is on the same side of the equation with the other terms, it has been observed, that it is always equal to the continual product of all the roots, with their signs changed; consequently q is equal to the product of all the roots under their own signs, when that quantity is on the other or right-hand side of the equation: but the product of the two less roots is always positive, because they are of the same affection, either both + or both −; and therefore this product, drawn into the third or greatest root, will generate another product, equal to q, and of the same affection with this root.

7. But the roots of equations of the above form, are not only positive, negative, or nothing, but sometimes also imaginary. We have found that the greatest root is positive when q is positive, and negative when q is negative; as also, that one root is = to 0 when q is = 0, and in this case the other two roots must be equal to each other, with contrary signs. But to discover the cases in which the equation has imaginary roots, as well as many other properties of the equation, it will be proper to consider the generation of it as follows.

8. The roots of equations becoming imaginary in pairs, the number of imaginary roots is always even; and therefore the cubic equation has either two imaginary roots, or

none at all; and consequently it has at least always one real root. Let that root be represented by r, which may be either positive or negative, and may be any one of the real roots, when none of them are imaginary: then, since any one of the roots is equal to the sum of the other two, with their signs changed, the other two roots may be represented by $-\frac{1}{2}r \pm$ some other quantity, since the sum of these two, with the signs changed, is $= r$. Now this supplemental quantity, which is to be connected with $-\frac{1}{2}r$, by the signs $+$ and $-$, to compose the other two roots, will be a real quantity when those roots are real, but an imaginary one when they are imaginary, since the other part, $-\frac{1}{2}r$, of those two roots, is real, by the hypothesis. Let this supplemental quantity be represented by e, when it is real, or $e\sqrt{-1}$ or $\sqrt{-e^2}$, when it is imaginary: we shall use the quantity e in what follows for the real roots; and it is evident, that by changing e for $e\sqrt{-1}$, or e^2 for $-e^2$; that is, by barely changing the sign of e^2 wherever it is found, the expressions will become adapted to the imaginary roots. Hence then, the three roots are represented by r, and $-\frac{1}{2}r + e$, and $-\frac{1}{2}r - e$; and consequently the three equations, from whose continual multiplication by one another, the cubic equation is to be generated, will be $x - r = 0$, and $x + \frac{1}{2}r - e = 0$, and $x + \frac{1}{2}r + e = 0$.

9. Let now these three equations be multiplied together, and there will be produced this general cubic equation, wanting the second term, namely, $x^3 \begin{smallmatrix}-\frac{3}{4}r^2 \\ -e^2\end{smallmatrix}\, x - r(\frac{1}{4}r^2 - e^2) = 0$, or $x^3 \begin{smallmatrix}-\frac{3}{4}r^2 \\ -e^2\end{smallmatrix}\, x = r\,(\frac{1}{4}r^2 - e^2)$, having three real roots; and if the sign of e^2 be changed from $-$ to $+$, it will then represent all the cases which have only one real, and two imaginary roots: and from the bare inspection of this equation the following properties are easily drawn.

10. First, we hence find, that when the equation has three real roots, the sign of the second term is always $-$; for the coefficient of that term, or p, is $= -\frac{3}{4}r^2 - e^2$, which is always negative, when r and e are real quantities. And

consequently, when p is positive, the equation has two imaginary roots, since $-p$ includes all the cases of three real roots. But it does not therefore follow, that when p is negative, the three roots are always real; and indeed there are imaginary roots not only whenever p is positive, but sometimes also when p is negative: for since p is $= -\frac{3}{4}r^2 - e^2$ in all the cases of three real roots, it will be $p = -\frac{3}{4}r^2 + e^2$ for all the cases of two imaginary roots; and it is evident, that p will be either positive or negative, according as e^2 is greater or less than $\frac{3}{4}r^2$.

11. But to find the cases of $-p$ when the roots are all real, and when not, will require some further consideration: and in order to that, it must be observed, that e^2 ought to be positive, and less than $\frac{3}{4}r^2$; but the limit between the cases of real and imaginary roots, is when $e^2 = 0$, or $e = 0$; and then p becomes $= -\frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$; consequently then $(\frac{1}{3}p)^3 = (\frac{1}{4}r^2)^3 = \frac{1}{64}r^6$, which is $= (\frac{1}{2}q)^2 = (\frac{1}{8}r^3)^2 = \frac{1}{64}r^6$, that is, when e is $= 0$, then $(\frac{1}{3}p)^3$ is $= (\frac{1}{2}q)^2$; and consequently, when $(\frac{1}{3}p)^3$ is less than $(\frac{1}{2}q)^2$, the equation has two imaginary roots; otherwise none, the sign of p being $-$. Thus then we easily perceive, in all cases, the nature of the roots, as to real and imaginary; namely, partly from the sign of p, and partly from the relation of p to q: for the equation has always two imaginary roots when p is positive; it has also two imaginary roots when p is negative, and $(\frac{1}{3}p)^3$ less than $(\frac{1}{2}q)^2$; in the other case, the roots are all real, namely, when p is negative and $(\frac{1}{3}p)^3$ either equal to or greater than $(\frac{1}{2}q)^2$.

12. Further, when p is $= 0$, the equation has two imaginary roots; for this cannot happen but by $-e^2$ becoming $+e^2$, in the value of p, and $=$ to $\frac{3}{4}r^2$; and then $p = -\frac{3}{4}r^2 + e^2 = -\frac{3}{4}r^2 + \frac{3}{4}r^2 = 0$, and $q = r(\frac{1}{4}r^2 + e^2) = r(\frac{1}{4}r^2 + \frac{3}{4}r^2) = r \cdot r^2 = r^3$; and consequently the above general equation becomes barely $x^3 = r^3$, which therefore, besides one real root $x = r$, has also two imaginary roots.

13. Hence also it again appears, that the greatest root is always of the same affection, in respect of positive and ne-

gative, with q on the right-hand side of the equation, these being either both positive or both negative together; and the other two roots of the contrary sign. For if r be the greatest root, then is $\frac{1}{2}r$ greater than e, and $\frac{1}{4}r^2$ greater than e^2, and $\frac{1}{4}r^2 - e^2$ always positive, and consequently the product $r\,(\frac{1}{4}r^2 - e^2)$, or q, will have the same sign with r. But if r be one of the less roots, the contrary of this will happen; for then $\frac{1}{2}r$ is less than e, and consequently $\frac{1}{4}r^2$ less than e^2, and so $\frac{1}{4}r^2 - e^2$ a negative quantity, and therefore the product $r\,(\frac{1}{4}r^2 - e^2)$, or q, will have the sign contrary to that of r; that is, q and the less roots have different signs, and consequently q and the greatest root the same sign, since the sign of the greatest root is always contrary to that of the other two roots.

14. Again, when q or $r\,(\frac{1}{4}r^2 - e^2)$ is positive, then r denotes the greatest root; for then $\frac{1}{4}r^2$ is greater than e^2, or $\frac{1}{2}r$ greater than e, and r greater than either $-\frac{1}{2}r + e$ or $-\frac{1}{2}r - e$. But when q or $r\,(\frac{1}{4}r^2 - e^2)$ is negative, then r represents one of the other two roots in the equation; since then e is greater than $\frac{1}{2}r$, and $-\frac{1}{2}r - e$ greater than r. Lastly, when q is between the positive and negative states, or $q = 0$, then r ought to be neither the greatest nor one of the less roots, if we may so speak, that is, two of the roots are equal, and the third root $= 0$, since then $\frac{1}{4}r^2$ must be $= e^2$, or $\frac{1}{2}r = e$.

15. Hence it appears, then in general, that the sign of p determines the nature of the roots as to real and imaginary, and the sign of q determines the affection of the roots as to positive and negative. Let us illustrate these rules by a few examples.

16. The equation $x^3 - 9x = 10$ has all its three roots real, because $p = -9$ is negative, and $(\frac{1}{3}p)^3 = 3^3 = 27$ is greater than $(\frac{1}{2}q)^2$, which is $= 5^2 = 25$; and the greatest of the roots is positive, because $q = 10$ is positive; and the two less roots negative.

17. The equation $x^3 - 9x = -10$ has the same three real roots as the former, but with the contrary signs, the sign of

the greatest root being now negative, because $q = -10$ is negative.

18. But the equation $x^3 + 9x = \pm 10$ has only one real root, and two imaginary roots, because $p = 9$ is positive; and the sign of the real root is + or − according as the sign of q or 10 is + or −.

19. The equation $x^3 + 6x = \pm 10$ has also two imaginary roots, and one real root, which is + or − as it is + 10 or − 10, for the same reason as before.

20. The equation $x^3 - 6x = \pm 10$ has also two imaginary roots, because $(\frac{1}{3}p)^3 = 2^3 = 8$, is less than $(\frac{1}{2}q)^2 = 5^2 = 25$.

21. But the equation $x^3 - 12x = \pm 16$ has all its roots real, because $(\frac{1}{3}p)^3 = 4^3 = 64$, is $= (\frac{1}{2}q)^2 = 8^2 = 64$.

22. And the equation $x^3 + 12x = \pm 16$ has only one real root, because $p = +12$ is positive.

23. Let us now consider the other properties and relations of the roots, arising from certain assumed relations between e and r, and from considering e either as real, imaginary, or nothing, that is e^2 as positive, negative, or nothing.

24. When e is a real quantity, the general equation is $x^3 {-\frac{3}{4}r^2 \atop -e^2} x = r(\frac{1}{4}r^2 - e^2)$, and all the roots are real.

25. When e is imaginary, the general equation is $x^3 {-\frac{3}{4}r^2 \atop +e^2} x = r(\frac{1}{4}r^2 + e^2)$, and two of the roots are imaginary.

26. When e is between these two states, or $= 0$, the equation becomes $x^3 - \frac{3}{4}r^2 x = \frac{1}{4}r^3$, and the root $r = \sqrt{\frac{4}{3}p} = \sqrt[3]{4q} = \frac{3q}{p}$; for in this case $p = \frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$. Also the other two roots, $-\frac{1}{2}r \pm e$, are each $= -\frac{1}{2}r$.

27. Assume now any general relation, between the root r, and the supplemental part e, of the other two roots, as suppose $r^2 : e^2 :: 4 : n$, or $e^2 = \frac{1}{4}nr^2$, or $e = \frac{1}{2}r\sqrt{n}$, where n represents either nothing, or any quantity, whether positive or negative, viz, positive when e and all the three roots are real, or negative when e and two of the roots are imaginary. Substitute now $\frac{1}{4}nr^2$ instead of e^2, in the general equation

$x^3 \begin{matrix} -\frac{3}{4}r^2 \\ -e^2 \end{matrix} x = r\,(\frac{1}{4}r^2 - e^2)$, and that equation will become $x^3 - \frac{3+n}{4}r^2 x = \frac{1-n}{4}r^3$. Here then $p = \frac{3+n}{4}r^2$, and $q = \frac{1-n}{4}r^3$, and consequently the root $r = \sqrt{\frac{4p}{3+n}} = \sqrt[3]{\frac{4q}{1-n}} = \frac{n+3}{n-1} \cdot \frac{q}{p}$, expressed in three different ways. The other roots, the general values of which are $-\frac{1}{2}r \pm e$, become $-\frac{1}{2}r \pm \sqrt{\frac{1}{4}nr^2} = -\frac{1}{2}r \pm \frac{1}{2}r\sqrt{n} = -\frac{1}{2}r \times (1 \pm \sqrt{n})$.

28. Hence then, in an easy and general manner, we can represent any form or case of the general equation, with all the circumstances of the roots, by only taking, in these last formulæ, any particular number for n, either positive or negative, integral or fractional, &c. As, if $n = 1$; then the equation becomes $x^3 - r^2x = \frac{0}{4}r^3$, or $= 0$, the value of $e = \frac{1}{2}r$, the root $r = \sqrt{p} = \sqrt[3]{\frac{4q}{0}} = \frac{4q}{0p}$, and the other two roots $= -\frac{1}{2}r\,(1 \pm \sqrt{1}) = -\frac{1}{2}r \cdot 2$ and $-\frac{1}{2}r \cdot 0 = -r$ and 0.

29. If $n = -1$, the equation will be $x^3 - \frac{1}{2}r^2\,x = \frac{1}{2}r^3$, the value of $e = \frac{1}{2}r\sqrt{-1}$, the root $r = \frac{q}{p} = \sqrt{2p} = \sqrt[3]{2q}$, and the other two roots $= -\frac{1}{2}r\,(1 \pm \sqrt{1})$, imaginary.

30. And thus, by taking several different values of n, positive and negative, the various corresponding circumstances and relations, of the equation and roots, will be as exhibited in the following table.

Forms of cases.	Values of n.	Values of e.	Forms of the equation.	Values of the root r, viz. $r=$	Values of the two other roots, viz. $-\frac{1}{2}r\times$
	$+n$	$\frac{1}{2}r\sqrt{+n}$	$x^3-\frac{n+3}{4}r^2x=-\frac{n-1}{4}r^3$	$\frac{n+3.q}{n-1.p}=\sqrt{\frac{4p}{n+3}}=\sqrt[3]{\frac{4q}{n-1}}$	$1\pm\sqrt{+n}$
1	$+12$	$\frac{1}{2}r\sqrt{+12}$	$x^3-\frac{15}{4}r^2x=-\frac{11}{4}r^3$	$\frac{15q}{11p}=\sqrt{\frac{4p}{15}}=\sqrt[3]{\frac{4q}{11}}$	$1\pm\sqrt{+12}$
2	$+11$	$\frac{1}{2}r\sqrt{+11}$	$x^3-\frac{14}{4}r^2x=-\frac{10}{4}r^3$	$\frac{14q}{10p}=\sqrt{\frac{4p}{14}}=\sqrt[3]{\frac{4q}{10}}$	$1\pm\sqrt{+11}$
3	$+10$	$\frac{1}{2}r\sqrt{+10}$	$x^3-\frac{13}{4}r^2x=-\frac{9}{4}r^3$	$\frac{13q}{9p}=\sqrt{\frac{4p}{13}}=\sqrt[3]{\frac{4q}{9}}$	$1\pm\sqrt{+10}$
4	$+9$	$\frac{1}{2}r\sqrt{+9}$	$x^3-\frac{12}{4}r^2x=-\frac{8}{4}r^3$	$\frac{12q}{8p}=\sqrt{\frac{4p}{12}}=\sqrt[3]{\frac{4q}{8}}$	$1\pm\sqrt{+9}$
5	$+8$	$\frac{1}{2}r\sqrt{+8}$	$x^3-\frac{11}{4}r^2x=-\frac{7}{4}r^3$	$\frac{11q}{7p}=\sqrt{\frac{4p}{11}}=\sqrt[3]{\frac{4q}{7}}$	$1\pm\sqrt{+8}$
6	$+7$	$\frac{1}{2}r\sqrt{+7}$	$x^3-\frac{10}{4}r^2x=-\frac{6}{4}r^3$	$\frac{10q}{6p}=\sqrt{\frac{4p}{10}}=\sqrt[3]{\frac{4q}{6}}$	$1\pm\sqrt{+7}$
7	$+6$	$\frac{1}{2}r\sqrt{+6}$	$x^3-\frac{9}{4}r^2x=-\frac{5}{4}r^3$	$\frac{9q}{5p}=\sqrt{\frac{4p}{9}}=\sqrt[3]{\frac{4q}{5}}$	$1\pm\sqrt{+6}$
8	$+5$	$\frac{1}{2}r\sqrt{+5}$	$x^3-\frac{8}{4}r^2x=-\frac{4}{4}r^3$	$\frac{8q}{4p}=\sqrt{\frac{4p}{8}}=\sqrt[3]{\frac{4q}{4}}$	$1\pm\sqrt{+5}$
9	$+4$	$\frac{1}{2}r\sqrt{+4}$	$x^3-\frac{7}{4}r^2x=-\frac{3}{4}r^3$	$\frac{7q}{3p}=\sqrt{\frac{4p}{7}}=\sqrt[3]{\frac{4q}{3}}$	$1\pm\sqrt{+4}$
10	$+3$	$\frac{1}{2}r\sqrt{+3}$	$x^3-\frac{6}{4}r^2x=-\frac{2}{4}r^3$	$\frac{6q}{2p}=\sqrt{\frac{4p}{6}}=\sqrt[3]{\frac{4q}{2}}$	$1\pm\sqrt{+3}$
11	$+2$	$\frac{1}{2}r\sqrt{+2}$	$x^3-\frac{5}{4}r^2x=-\frac{1}{4}r^3$	$\frac{5q}{1p}=\sqrt{\frac{4p}{5}}=\sqrt[3]{\frac{4q}{1}}$	$1\pm\sqrt{+2}$
12	$+1$	$\frac{1}{2}r\sqrt{+1}$	$x^3-\frac{4}{4}r^2x=\mp\frac{0}{4}r^3$	$\frac{4q}{0p}=\sqrt{\frac{4p}{4}}=\sqrt[3]{\frac{4q}{0}}$	$1\pm\sqrt{+1}$
13	±0	$\frac{1}{2}r\sqrt{\pm0}$	$x^3-\frac{3}{4}r^2x=+\frac{1}{4}r^3$	$\frac{3q}{1p}=\sqrt{\frac{4p}{3}}=\sqrt[3]{\frac{4q}{1}}$	$1\pm\sqrt{\pm0}$
14	-1	$\frac{1}{2}r\sqrt{-1}$	$x^3-\frac{2}{4}r^2x=+\frac{2}{4}r^3$	$\frac{2q}{2p}=\sqrt{\frac{4p}{2}}=\sqrt[3]{\frac{4q}{2}}$	$1\pm\sqrt{-1}$
15	-2	$\frac{1}{2}r\sqrt{-2}$	$x^3-\frac{1}{4}r^2x=+\frac{3}{4}r^3$	$\frac{1q}{3p}=\sqrt{\frac{4p}{1}}=\sqrt[3]{\frac{4q}{3}}$	$1\pm\sqrt{-2}$
16	-3	$\frac{1}{2}r\sqrt{-3}$	$x^3\mp\frac{0}{4}r^2x=+\frac{4}{4}r^3$	$\frac{0q}{4p}=\sqrt{\frac{4p}{0}}=\sqrt[3]{\frac{4q}{4}}$	$1\pm\sqrt{-3}$
17	-4	$\frac{1}{2}r\sqrt{-4}$	$x^3+\frac{1}{4}r^2x=+\frac{5}{4}r^3$	$\frac{1q}{5p}=\sqrt{\frac{4p}{1}}=\sqrt[3]{\frac{4q}{5}}$	$1\pm\sqrt{-4}$
18	-5	$\frac{1}{2}r\sqrt{-5}$	$x^3+\frac{2}{4}r^2x=+\frac{6}{4}r^3$	$\frac{2q}{6p}=\sqrt{\frac{4p}{2}}=\sqrt[3]{\frac{4q}{6}}$	$1\pm\sqrt{-5}$
19	-6	$\frac{1}{2}r\sqrt{-6}$	$x^3+\frac{3}{4}r^2x=+\frac{7}{4}r^3$	$\frac{3q}{7p}=\sqrt{\frac{4p}{3}}=\sqrt[3]{\frac{4q}{7}}$	$1\pm\sqrt{-6}$
20	-7	$\frac{1}{2}r\sqrt{-7}$	$x^3+\frac{4}{4}r^2x=+\frac{8}{4}r^3$	$\frac{4q}{8p}=\sqrt{\frac{4p}{4}}=\sqrt[3]{\frac{4q}{8}}$	$1\pm\sqrt{-7}$
21	-8	$\frac{1}{2}r\sqrt{-8}$	$x^3+\frac{5}{4}r^2x=+\frac{9}{4}r^3$	$\frac{5q}{9p}=\sqrt{\frac{4p}{5}}=\sqrt[3]{\frac{4q}{9}}$	$1\pm\sqrt{-8}$
22	-9	$\frac{1}{2}r\sqrt{-9}$	$x^3+\frac{6}{4}r^2x=+\frac{10}{4}r^3$	$\frac{6q}{10p}=\sqrt{\frac{4p}{6}}=\sqrt[3]{\frac{4q}{10}}$	$1\pm\sqrt{-9}$
23	-10	$\frac{1}{2}r\sqrt{-10}$	$x^3+\frac{7}{4}r^2x=+\frac{11}{4}r^3$	$\frac{7q}{11p}=\sqrt{\frac{4p}{7}}=\sqrt[3]{\frac{4q}{11}}$	$1\pm\sqrt{-10}$
24	-11	$\frac{1}{2}r\sqrt{-11}$	$x^3+\frac{8}{4}r^2x=+\frac{12}{4}r^3$	$\frac{8q}{12p}=\sqrt{\frac{4p}{8}}=\sqrt[3]{\frac{4q}{12}}$	$1\pm\sqrt{-11}$
25	-12	$\frac{1}{2}r\sqrt{-12}$	$x^3+\frac{9}{4}r^2x=+\frac{13}{4}r^3$	$\frac{9q}{13p}=\sqrt{\frac{4p}{9}}=\sqrt[3]{\frac{4q}{13}}$	$1\pm\sqrt{-12}$
	$-n$	$\frac{1}{2}r\sqrt{-n}$	$x^3+\frac{n-3}{4}r^2x=+\frac{n+1}{4}r^3$	$\frac{n-3.q}{n+1.p}=\sqrt{\frac{4p}{n-2}}=\sqrt[3]{\frac{4q}{n+1}}$	$1\ \sqrt{-n}$

31. From the bare inspection of this table, several useful and curious observations may be made. And first it appears, that when q is positive, as in all the forms after the 12th, r is the greatest root; but when q is negative, or in all the cases to the 12th, r is one of the less roots.

32. In all cases before the 4th form, r is the least root, because $\frac{\sqrt{10}-1}{2}$, or $\frac{\sqrt{11}-1}{2}$, &c, is always greater than 1; and in all such forms, $(\frac{1}{2}q)^2$ is less than $(\frac{1}{3}p)^3$; but the former approaches nearer and nearer to an equality with the latter, till the 4th form, where $(\frac{1}{2}q)^2$ is become $= (\frac{1}{3}p)^3$, and r is then equal to one of the other roots, because $\frac{\sqrt{9}-1}{2} = \frac{2}{2} = 1$.

33. From hence r becomes the middle root, and continues so to the 12th form, where it becomes equal to what has hitherto been the greatest root, and the other root becomes at this place $= 0$; and $(\frac{1}{2}q)^2$ has decreased from the 4th form, all the way more and more, in respect of $(\frac{1}{3}p)^3$, till at this 12th form it has become $= 0$, or infinitely less than $(\frac{1}{3}p)^3$.

34. From this place, r becomes the greatest root, the sign of q changes to $+$, and $(\frac{1}{2}q)^3$ again increases in respect of $(\frac{1}{3}p)^3$, till at the 13th case it becomes again equal to it, and the two less roots equal to each other, like as at the 4th form.

35. From hence $(\frac{1}{2}q)^2$ becomes greater than $(\frac{1}{3}p)^3$, and increases more and more in respect of it, till at the 16th step, where p is $= 0$, or $(\frac{1}{2}q)^2$ infinitely greater than $(\frac{1}{3}p)^3$.

36. From this place the sign of p becomes $+$, and $(\frac{1}{2}q)^2$ continually decreases in respect of $(\frac{1}{3}p)^3$, to infinity.

37. By help of this table, we may find the roots of any cubic equation $x^3 \mp px = q$, whenever we can assign the relation between $\sqrt{p}$ and $\sqrt[3]{q}$. For since one root r is always $= \frac{(n \pm 3)q}{(n \mp 1)p} = \sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, and the other two roots $= -\frac{1}{2}r(1 \pm \sqrt{\pm n})$, it follows, that if, from the equation

$\sqrt{\frac{4p}{n\pm3}} = \sqrt[3]{\frac{4q}{n\mp1}}$, where the two denominators under the radicals differ by 4, we can assign the value of n, the above formula will give us the roots.

38. As, if the equation be $x^3 - 18x = -27$. Here $p = 18$, and $q = 27$; then $\sqrt{\frac{4p}{8}} = \sqrt{\frac{p}{2}} = \sqrt{9} = 3$, and $\sqrt[3]{\frac{4q}{4}} = \sqrt[3]{27} = 3$ also; therefore $n + 3 = 8$, or $n - 1 = 4$, either of which gives $n = 5$: consequently, $r = \frac{(n+3)q}{(n-1)p} = \frac{8q}{4p} = \frac{2q}{p} = \frac{54}{18} = 3$, is the middle root, because $\frac{8q}{4p}$ is found between the 4th and 12th cases, which are the limits of the middle roots: and $-\frac{1}{2}r(1 \pm \sqrt{n}) = -\frac{3}{2}(1 \pm \sqrt{5}) = 4{\cdot}854102$ and $1{\cdot}854102$, are the greatest and least roots. Or, these two roots may be also found in the same manner from the table of forms, which contains all the roots of every equation, thus: by a few trials we find $\sqrt{\frac{4p}{20{\cdot}95}} = \sqrt[3]{\frac{4q}{16{\cdot}95}}$ nearly, and therefore $\frac{20{\cdot}95q}{16{\cdot}95p} = 1{\cdot}854$ is the least root, because here $n = 17{\cdot}95$, which lies far above the limit for the least roots, which is at the 4th form, where n is $= 9$. And lastly, $\sqrt{\frac{4p}{3{\cdot}0557}} = \sqrt[3]{\frac{4q}{{\cdot}9443}}$ nearly, and therefore $\frac{3{\cdot}0557q}{{\cdot}9443p} = 4{\cdot}854$ is the greatest root, because $\frac{3{\cdot}0557q}{{\cdot}9443p}$ is found between the 12th and 13th forms, which are the limits between which lies the greatest root, of every equation that has all its roots real.

39. Again, let the equation be $x^3 + 2x = 12$. Here $p = 2$, and $q = 12$; hence $\sqrt{\frac{4p}{2}} = \sqrt{2p} = \sqrt{4} = 2$, and $\sqrt[3]{\frac{4q}{6}} = \sqrt[3]{\frac{2}{3}q} = \sqrt[3]{8} = 2$ also; therefore $n - 3 = 2$, or $n + 1 = 6$, either of which gives $n = 5$. Consequently $r = \frac{(n-3)q}{(n+1)p} = \frac{2q}{6p} = \frac{q}{3p} = \frac{12}{6} = 2$, and the other two roots are $-\frac{1}{2}r(1 \pm \sqrt{-n}) = -1(1 \pm \sqrt{-5}) = -1 \mp \sqrt{-5}$.

40. But it is only by trials that we find out a proper value for n in such cases as these; and this is perhaps at-

tended with no less trouble, than the searching out one of the roots by trials, from the original cubic equation itself. This method of finding the roots would indeed be effectual and satisfactory, if we had a direct method of determining the value of n, from the equation $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, by an equation under the 3d degree; but by reducing this equation out of radicals, there results another cubic equation, of no less difficulty to resolve than the original one. We must therefore search for other methods of determining the roots; and first it will be proper to treat of the rule which is called Cardan's.

41. Let $x^3 + px = q$ be the general equation, where p and q denote any given numbers with their signs, positive or negative. And let $z + y$ denote one of the roots of this equation, that is, let the root be divided into any two parts z and y. Hence then $x = z + y$; which value of x being substituted for it, in the original equation $x^3 + px = q$, that equation will become $z^3 + 3z^2y + 3zy^2 + y^3 + p(z + y) = q$, or $z^3 + y^3 + 3zy(z + y) + p(z + y) = q$. Now, on introducing the two unknown quantities z and y, we supposed only one condition or equation, namely, $z + y = x$; we are therefore yet at liberty to assume any other possible condition we please: but this other condition ought to be such as will make the equation reducible to a simple one, or to a quadratic, in order to obtain from it the value of z or y: and for this purpose there does not seem to be any other proper condition, beside that which supposes $3zy$ to be $= -p$; and in consequence of this supposition, the equation becomes barely $z^3 + y^3 = q$. Now, from the square of this equation, let 4 times the cube of $zy = -\frac{1}{3}p$ be subtracted, and there will remain $z^6 - 2z^3y^3 + y^6 = q^2 + \frac{4}{27}p^3$, the square root of which is $z^3 - y^3 = \sqrt{(q^2 + \frac{4}{27}p^3)}$; this last being added to, and subtracted from, the equation $z^3 + y^3 = q$,

$$\text{gives} \begin{cases} 2z^3 = q + \sqrt{(q^2 + \frac{4}{27}p^3)} = q + 2\sqrt{[(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3]}, \\ 2y^3 = q - \sqrt{(q^2 + \frac{4}{27}p^3)} = q - 2\sqrt{[(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3]}, \end{cases}$$

hence dividing by 2, and extracting the cube roots, we have

$$\begin{cases} z = \sqrt[3]{(\tfrac{1}{2}q + \sqrt{[(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3]})} \times 1 \text{ or} \times -\frac{1\pm\sqrt{-3}}{2} \\ y = \sqrt[3]{(\tfrac{1}{2}q - \sqrt{[(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3]})} \times 1 \text{ or} \times -\frac{1\mp\sqrt{-3}}{2} \end{cases},$$

the three values of z and y; for every quantity has three different forms of the cube root, and the cube root of 1, is not only 1, but also $-\frac{1+\sqrt{-3}}{2}$ or $-\frac{1-\sqrt{-3}}{2}$. Hence then the three values of $z+y$ or x, or the three roots of the equation $x^3 + px = q$, are

$$\sqrt[3]{(\tfrac{1}{2}q+\sqrt{[(\tfrac{1}{2}q)^2+(\tfrac{1}{3}p)^3]})} \times 1 \text{ or } \times -\frac{1+\sqrt{-3}}{2} \text{ or } \times -\frac{1-\sqrt{-3}}{2} +$$
$$\sqrt[3]{(\tfrac{1}{2}q-\sqrt{[(\tfrac{1}{2}q)^2+(\tfrac{1}{3}p)^3]})} \times 1 \text{ or } \times -\frac{1-\sqrt{-3}}{2} \text{ or } \times -\frac{1+\sqrt{-3}}{2},$$

where the signs of $\sqrt{-3}$ must be opposite, in the values of z and y, that is, when it is $\frac{1\pm\sqrt{-3}}{2}$ in the one, it must be $\frac{1\mp\sqrt{-3}}{2}$ in the other, otherwise their product zy will not be $= -\frac{1}{3}p$, as it ought to be.

42. Or if we put $a = \frac{1}{3}p$, and $b = \frac{1}{2}q$, the same three roots will be

$\sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]} =$ the 1st root or r,

$-\frac{1}{2}\sqrt[3]{[b + \sqrt{(b^2+a^3)}]}.(1-\sqrt{-3})$
$\quad -\frac{1}{2}\sqrt[3]{[b - \sqrt{(b^2+a^3)}]}.(1+\sqrt{-3})$ the 2d root.

$-\frac{1}{2}\sqrt[3]{[b + \sqrt{(b^2+a^3)}]}.(1+\sqrt{-3})$
$\quad -\frac{1}{2}\sqrt[3]{[b - \sqrt{(b^2+a^3)}]}.(1-\sqrt{-3})$ the 3d root.

43. Or again, the 1st root r being

$\sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]}$, the other two will be

$-\frac{1}{2}r + \frac{\sqrt{-3}}{2}\sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} - \frac{\sqrt{-3}}{2}\sqrt[3]{[b - \sqrt{(b^2 + a^2)}]} =$ the 2d root, and

$-\frac{1}{2}r - \frac{\sqrt{-3}}{2}\sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \frac{\sqrt{-3}}{2}\sqrt[3]{[b - \sqrt{(b^2 + a^3)}]} =$ the 3d root.

44. Or, if we put $s = \sqrt[3]{[b + \sqrt{(b^2 + a^3)}]}$, and $d = \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]}$, the roots will be

$s + d = r$ the 1st root,

$-\frac{s+d}{2} + \frac{s-d}{2}\sqrt{-3} =$ the 2d root,

$-\frac{s+d}{2} - \frac{s-d}{2}\sqrt{-3} =$ the 3d root.

45. The first of these roots, x or r, $= s + d = \sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]}$, is that which is called Cardan's rule, by whom it was first published, but invented by Ferreus. And this is always a real root, though it is not always the greatest root, as it has been commonly thought to be.

46. The first root $r = s + d = \sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]}$, though it be always a real quantity, yet often assumes an imaginary form, when particular numbers are substituted instead of the letters a and b, or p and q. And this it is evident will happen whenever a is negative, and a^3 greater than b^2, or $(\frac{1}{3}p)^3$ greater than $(\frac{1}{2}q)^2$; for then $\sqrt{(b^2 + a^3)}$ becomes $\sqrt{(b^2 - a^3)}$, the square root of a negative quantity, which is imaginary. And this will evidently happen whenever the equation has three real roots, but at no time else, that is in all the first 13 cases of the foregoing table, where $(\frac{1}{3}p)^3$ is greater than $(\frac{1}{2}q)^2$, and p negative; the 4th and 13th only excepted, when $(\frac{1}{3}p)^3$ is $= (\frac{1}{2}q)^2$, and therefore $\sqrt{(b^2 - a^3)} = 0$, and two of the roots become equal, but with contrary signs. This root can never assume an imaginary form when a or p is positive, nor yet when p is negative and $(\frac{1}{2}q)^2$ greater than $(\frac{1}{3}p)^3$; for in both these cases the quantity $\sqrt{(b^2 \pm a^3)}$ is real, or the square root of a positive quantity. And these take place after the first 13 cases of the table of forms, that is, in all the cases which have only one real root. So that this rule of Cardan's always gives the root in an imaginary form when the equation has no imaginary roots, but in the form of a real quantity when it has imaginary roots.

47. It may perhaps seem wonderful, that Cardan's theorem should thus exhibit the root of an equation under the form of an imaginary or impossible quantity, always when

the equation has no imaginary roots, but at no time else; and it may justly be demanded what can be the reason of so curious an accident. But this seeming paradox will be cleared up by the following consideration. It is plain, that this circumstance must have happened either through some impropriety in the manner of deducing the values of z and y, from the two assumed equations $x = z + y$, and $zy = -\frac{1}{3}p$, or else by some impossibility in one of these two conditions themselves: but, on examination, the deductions are found to be all fairly drawn, and the operations rightly performed. The true cause must therefore lie concealed in one of these two conditions $x = z + y$ and $zy = -\frac{1}{3}p$. In the first of them it cannot be, because it only supposes that a quantity x can be divided into two parts z and y, which is evidently a possible supposition: it can therefore no where exist but in the latter, namely, $zy = -\frac{1}{3}p$. Now this supposition is this, that the product of the two parts z and y, into which the constant quantity x is divided, is equal to $\frac{1}{3}p$ with its sign changed. Now this may always take place when p is positive; for then, $-\frac{1}{3}p$ will be negative, and two numbers, the one positive and the other negative, may always be taken such, that their product shall be equal to any negative number whatever, and yet their sum be equal to a given quantity x; and this is done by taking the positive one as much greater than x, as the other is negative; for thus it is evident the positive and negative numbers may be increased without end: there is no impossibility then in the supposition when p is positive; and therefore then the formula ought to exhibit only real quantities, that is, in all the cases after the 16th in the table of forms, as we have before found. But the same thing cannot always happen when p is negative, or $-\frac{1}{3}p = zy$ is positive: for that zy may be positive, the signs of the two factors z and y must be alike, either both $+$ or both $-$, that is, both $+$ when the sign of x is $+$, or both $-$ when that is $-$: but it is well known, that the greatest product which can be made of the two parts, into which a constant quantity x may be divided, is

when the parts are equal to each other, or each equal $\frac{1}{2}x$, and therefore the greatest product is equal to $(\frac{1}{2}x)^2$ or $\frac{1}{4}x^2$: therefore, if $\frac{1}{4}x^2$ be equal to or greater than $-\frac{1}{3}p$, the condition which supposes that xy is $= -\frac{1}{3}p$, is possible, and the formula ought to express the root by real quantities only; otherwise not: but $\frac{1}{4}x^2$, or $\frac{1}{4}r^2$, which is the same thing, is always less than $-\frac{1}{3}p$ in the first 13 cases of the table of forms; and therefore, in all these cases, which are those in which $(\frac{1}{3}p)^3$ is greater than $(\frac{1}{2}q)^2$, or all those which have three real roots, the formula ought to exhibit the root with imaginary quantities, as we have before found to happen; the 4th and 13th cases only excepted, in which $(\frac{1}{3}p)^3$ is $= (\frac{1}{2}q)^2$, and therefore the quantity $\sqrt{(b^2 - a^3)}$ vanishes, and two of the roots are equal.

48. Thus then the real cause of this circumstance is made manifest, and it is found to be the necessary consequence of the arbitrary hypothesis that was made, which is found to be possible only in certain cases. So that we cannot expect the formula to exhibit a real quantity in the other cases, since an impossible hypothesis must needs lead to an absurd conclusion.

49. The other two roots $-\frac{s+d}{d} \pm \frac{s-d}{2}\sqrt{-3}$, in their general state, appear in an imaginary form; but on the substitution of numbers for the letters in any example, they come out real, or imaginary quantities, in those cases in which they ought to be such. For s being $= g + \sqrt{\mp h}$, and $d = g - \sqrt{\mp h}$, according as the roots are all real or only one is such; and $-\frac{s+d}{2} = -g = -\frac{1}{2}r$ always half the one real root, we have $\frac{s-d}{2} = \sqrt{\mp h}$ according to the said two cases; and consequently $\frac{s-d}{2}\sqrt{-3} = \sqrt{\pm 3h}$, a real or an imaginary quantity, according as the roots are to be real or imaginary.

50. The first root r being found from the formula $\sqrt[3]{[b + \sqrt{(b^2 + a^3)}]} + \sqrt[3]{[b - \sqrt{(b^2 + a^3)}]}$, or by any other

means, the other two roots may be exhibited in several other forms, besides the foregoing, as may be shown in the following manner.

51. The equation being $x^3 + px = q$, and one root r, by substitution we have $r^3 + pr = q$, and, by subtracting, it is - - - $x^3 - r^3 + p(x - r) = 0$, and, dividing by $x - r$, it becomes $x^2 + rx + r^2 + p = 0$.

Or this same equation may be found by barely dividing $x^3 + px - q = 0$ by $x - r = 0$, for the quotient is $x^2 + rx + r^2 + p = 0$. And the resolution of this quadratic equation gives $x = -\frac{1}{2}r \pm \sqrt{(-p - \frac{3}{4}r^2)} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{(-4p - 3r^2)}$, the other two roots. And hence again it appears, that these two roots are always imaginary, when p in the given equation is positive; as also when it is negative and less than $\frac{3}{4}r^2$; which again include all the cases of the table of forms after the 13th.

52. Again, since $r^3 + pr = q$, therefore $r^2 + p = \frac{q}{r}$, and $r^2 = -p + \frac{q}{r}$, and $-3r^2 = 3p - \frac{3q}{r}$; which being substituted in the above value of the two roots, they become $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{(-p - \frac{3q}{r})}$.

53. And again, if $-p$ be expelled from this last form, by means of its value $r^2 - \frac{q}{r}$, the same two roots will be expressed by $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{(r^2 - \frac{4q}{r})} = -\frac{1}{2}r \times [1 \pm \sqrt{(1 - \frac{4q}{r^3})}]$.

54. And further, if r^3 be expelled from this last form, by means of its value $q - pr$, the same two roots will also become $-\frac{1}{2}r \times [1 \pm \sqrt{(1 - \frac{4q}{q - pr})}] = -\frac{1}{2}r \times [1 \pm \sqrt{\frac{pr + 3q}{pr - q}}]$.

55. We might have derived the above forms in yet another manner, thus. The first root being r, let the other two roots be v and w: then we shall have these two equations, namely, $v + w = -r$, and $vwr = q$, or $vw = \frac{q}{r}$; from the square of the first of these, subtract 4 times the last, so shall $v^2 - 2vw + w^2 = r^2 - \frac{4q}{r}$; the root of this is

$v - w = \sqrt{(r^2 - \frac{4q}{r})}$; which being added to, and taken from $v + w = -r$, and dividing by 2, we have $\left.\begin{matrix} v \\ w \end{matrix}\right\} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{(r^2 - \frac{4q}{r})} = -\frac{1}{2}r \times [1 \pm \sqrt{(1 - \frac{4q}{r^3})}]$, the same with one of the formulæ above given; and then by substitution the others will be obtained.

56. To illustrate now the rules $x = s + d$, or $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, by some examples; suppose the given equation to be $x^3 - 36x = 91$. Here $p = -36$, $q = 91$, $a = \frac{1}{3}p = -12$, $b = \frac{91}{2}$; then $c = \sqrt{(b^2 + a^3)} = \sqrt{(\frac{8281}{4} - 1728)} = \sqrt{\frac{1369}{4}} = \frac{37}{2}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(\frac{91}{2} + \frac{37}{2})} = \sqrt[3]{64} = 4$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(\frac{91}{2} - \frac{37}{2})} = \sqrt[3]{27} = 3$. Consequently, $r = s + d = 4 + 3 = 7$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-7 \pm \sqrt{-3}}{2}$ the other two roots, which are imaginary.

57. Ex. 2. Let the equation be $x^3 + 30x = 117$. Here $a = \frac{1}{3}p = 10$, $b = \frac{1}{2}q = \frac{117}{2}$; then $c = \sqrt{(b^2 + a^3)} = \sqrt{(\frac{13689}{4} + 1000)} = \sqrt{\frac{17689}{4}} = \frac{133}{2}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(\frac{117}{2} + \frac{133}{2})} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(\frac{117}{2} - \frac{133}{2})} = \sqrt[3]{-\frac{16}{2}} = \sqrt[3]{-8} = -2$. Consequently, $r = s + d = 5 - 2 = 3$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-3 \pm 7\sqrt{-3}}{2}$ the other two roots, which are imaginary.

58. Ex. 3. If the equation be $x^3 + 18x = 6$, we shall have $a = 6$, and $b = 3$; then $c = \sqrt{(b^2 + a^3)} = \sqrt{(9 + 216)} = \sqrt{225} = 15$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(3 + 15)} = \sqrt[3]{18}$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(3 - 15)} = \sqrt[3]{-12} = -\sqrt[3]{12}$. Therefore $r = s + d = \sqrt[3]{18} - \sqrt[3]{12} = \cdot 331313$ the first root; and

$-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{18}-\sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18}+\sqrt[3]{12}}{2}\sqrt{-3}$ the other two roots.

59. Ex. 4. In the equation $x^3 - 15x = 4$, we have $a = -5$, $b = 2$; hence $c = \sqrt{(b^2 + a^3)} = \sqrt{(4 - 125)} = \sqrt{-121} = 11\sqrt{-1}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(2 + 11\sqrt{-1})} = 2 + \sqrt{-1}$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(2 - 11\sqrt{-1})} = 2 - \sqrt{-1}$. Therefore $r = s + d = 4$ the first root; and $-\frac{s+d}{2} \mp \frac{s-d}{2}\sqrt{-3} = -2 \pm \sqrt{-1} \,.\, \sqrt{-3} = -2 \pm \sqrt{3}$ the other two roots, which are also real.

60. Ex. 5. The equation $x^3 - 6x = 4$ gives $a = -2$, and $b = 2$; therefore $c = \sqrt{(b^2 + a^3)} = \sqrt{(4 - 8)} = \sqrt{-4} = 2\sqrt{-1}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(2 + 2\sqrt{-1})} = -1 + \sqrt{-1}$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(2 - 2\sqrt{-1})} = -1 - \sqrt{-1}$. And hence $r = s + d = -2$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-1} \,.\, \sqrt{-3} = 1 \pm \sqrt{3}$, which are the two extremes, or the greatest and least roots. So that in this example, Cardan's rule gives the middle root.

61. Ex. 6. Let the equation be $x^3 - 9x = -10$. Then $a = -3$ and $b = -5$; so that $c = \sqrt{(b^2 + a^3)} = \sqrt{(25 - 27)} = \sqrt{-2}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(-5 + \sqrt{-2})} = 1 + \sqrt{-2}$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(-5 - \sqrt{-2})} = 1 - \sqrt{-2}$. Hence $r = s + d = 2$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -1 \pm \sqrt{-2} \,.\, \sqrt{-3} = -1 \pm \sqrt{6}$ the greatest and least roots.

62. Ex. 7. Take the equation $x^3 - 12x = 9$. Here $a = -4$, and $b = \frac{9}{2}$; therefore $c = \sqrt{(b^2 + a^3)} = \sqrt{(\frac{81}{4} - 64)} = \sqrt{-\frac{175}{4}} = \frac{5}{2}\sqrt{-7}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(\frac{9}{2} + \frac{5}{2}\sqrt{-7})} = -\frac{3}{2} + \frac{1}{2}\sqrt{-7}$, and $d = \sqrt[3]{(b - c)} = \sqrt[3]{(\frac{9}{2} - \frac{5}{2}\sqrt{-7})} = -\frac{3}{2} - \frac{1}{2}\sqrt{-7}$. Hence $r = s + d = -3$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-7} \,.\, \sqrt{-3} = \frac{3 \pm \sqrt{21}}{2}$ the greatest and least roots.

63. Ex. 8. Again, from the equation $x^3 - 12x = -8\sqrt{2}$, we have $a = -4$, and $b = -4\sqrt{2}$; hence $c = \sqrt{(b^2 + a^3)} = \sqrt{(32 - 64)} = \sqrt{-32} = 4\sqrt{-2}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(-4\sqrt{2} + 4\sqrt{-2})} = \sqrt{2} + \sqrt{-2}$, and $d = \sqrt{2} - \sqrt{-2}$. So that $r = s + d = 2\sqrt{2}$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\sqrt{2} \pm \sqrt{-2} \,.\, \sqrt{-3} = -\sqrt{2} \pm \sqrt{6} = -\sqrt{2} \,.\, (1 \mp \sqrt{3})$ the greatest and least roots.

64. Ex. 9. But the equation $x^3 - 15x = 22$ gives $a = -5$, and $b = 11$; therefore $c = \sqrt{(b^2 + a^3)} = \sqrt{(121 - 125)} = \sqrt{-4}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(11 + \sqrt{-4})} = -1 - \sqrt{-4}$, and $d = -1 + \sqrt{-4}$. Consequently $r = s + d = -2$ the least root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-4} \,.\, \sqrt{-3} = 1 \pm \sqrt{12}$ the two greater roots.

65. Ex. 10. Further, in the equation $x^3 - 15x = 20$, we have $a = -5$, and $b = 10$; consequently $c = \sqrt{(b^2 + a^3)} = \sqrt{(100 - 125)} = \sqrt{-25} = 5\sqrt{-1}$, $s = \sqrt[3]{(b + c)} = \sqrt[3]{(10 + 5\sqrt{-1})}$, and $d = \sqrt[3]{(10 - 5\sqrt{-1})}$. Therefore $r = s + d = \sqrt[3]{(10 + 5\sqrt{-1})} + \sqrt[3]{(10 - 5\sqrt{-1})} =$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{(10+5\sqrt{-1})} + \sqrt[3]{(10-5\sqrt{-1})}}{2} \pm \frac{\sqrt[3]{(10+5\sqrt{-1})} - \sqrt[3]{(10-5\sqrt{-1})}}{2}\sqrt{-3} =$ the other two roots.

*65. Ex. 11. Lastly, taking the equation $x^3 - 7x = 6$. Here $a = -\frac{7}{3}$, and $b = 3$; therefore $c = \sqrt{(b^2 + a^3)} = \sqrt{(9 - \frac{343}{27})} = \sqrt{-\frac{100}{27}} = \frac{10}{9}\sqrt{-3}$; $s = \sqrt[3]{(b + c)} = \sqrt[3]{(3 + \frac{10}{9}\sqrt{-3})} = \frac{3}{2} + \frac{1}{6}\sqrt{-3}$; and $d = \frac{3}{2} - \frac{1}{6}\sqrt{-3}$; consequently, $r = s + d = 3$; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{3}{2} \pm \frac{1}{2} = -1$ and -2, the two less roots. So that all the three roots, in this example of the irreducible case, are rational.

66. Hence it appears, that Cardan's rule, $s + d$, brings out sometimes the greatest root, sometimes the middle root, and sometimes the least root.

Of the Roots by Infinite Series.

67. Another way of assigning the roots of a cubic equation, may be by infinite series, derived from the foregoing formulæ, namely, $s + d$ and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, or $\sqrt[3]{(b+c)} + \sqrt[3]{(b-c)}$ and
$-\frac{1}{2} \times [\sqrt[3]{(b+c)} + \sqrt[3]{(b-c)}] \pm \frac{1}{2}\sqrt{-3} \times [\sqrt[3]{(b+c)} - \sqrt[3]{(b-c)}]$.
For, by expanding $\sqrt[3]{(b \pm c)}$ in an infinite series, we shall evidently have all the roots expressed in such series.

68. Now $s = \sqrt[3]{(b+c)} = \sqrt[3]{b} \times : 1 + \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} + \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c,

and $d = \sqrt[3]{(b-c)} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c,

Hence $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4}$ &c,

for the first root, as it was found by Mr. Nicole, in the *Memoires de l'Acad.* 1738. Also
$s - d = \frac{2c}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4}$ &c. Therefore

$$\left.\begin{array}{l} -\frac{s+d}{2} \\ \pm \frac{s-d}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} -\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \text{ \&c,} \\ \pm\frac{c\sqrt{-3}}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \text{ \&c,} \end{array}\right.$$

for the other two roots, which were given by Clairaut, in his *Elemens d'Algebre.*

69. Hence again it appears, that when c^2 is positive, these two latter roots are imaginary; for then the factor $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}$ is imaginary. And that those roots are real when this c^2 is negative; for then this factor becomes $\frac{c\sqrt{-1} \times \sqrt{-3}}{\sqrt[3]{b^2}} = \frac{c\sqrt{3}}{\sqrt[3]{b^2}}$, a real quantity. But in this last case, the sign of every second term in the two series must be changed, namely, the signs of the terms containing the odd powers of the negative quantity c^2; for the series contain the letters as adapted to the positive sign only.

70. These series are proper for those cases only in which c^2 is not greater than b^2; for if c^2 were greater than b^2, they

would all diverge, and be of no use: and the series proper for the other cases, namely, in which c^2 is greater than b^2, we shall give below.

71. That c^2 be less than b^2, or the foregoing series be proper to be used, a or $\frac{1}{3}p$ must be a negative quantity; for if it be positive, then $c^2 = b^2 + a^3$ will be greater than b^2. But for this purpose a cannot be *any* negative quantity taken at pleasure; for if it be so taken, as that a^3 be greater than $2b^2$, then shall $-c^2 = a^3 - b^2$ be greater than b^2. And hence these series converge only in some of the cases of three real roots, and in some of those that have only one real root, namely, from the 16th form, to somewhere between the 12th and 13th forms, in the general table Art. 30, when b is positive, and consequently it includes some cases both with and without imaginary roots. But that in all the cases, the first series $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2}$ &c, is the greatest root, as will still more fully appear by consulting Art. 83.

72. Now, in the first place, when $a = 0$, or $c = b$, which is the limit, or 16th case in the table Art. 30, the equation being $x^3 = q = 2b$, then the only real root is $s = \sqrt[3]{(b + c)} = \sqrt[3]{2b} = \sqrt[3]{q} = \sqrt[3]{b} \times : 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} +$ &c. Hence also, dividing by $\sqrt[3]{b}$, we have $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c.

73. But in this case also the root is

$s + d = 2\sqrt[3]{b} \times : 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c. And consequently this is equal to the former series, or $2 \times : 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c $= 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}$ &c $= \sqrt[3]{2}$. Hence, by subtracting $1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c from both sides, we have

$1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c $= \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15}$ &c, which multiplied by $2\sqrt[3]{b}$, will also give the root of the same equation. And hence, adding $\frac{2}{3 \cdot 6} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c to both

sides of the last equation, we find that

$$1 \text{ is } = \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} + \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

Or, further, multiplying by 3, and subtracting 1, we have

$$2 = \frac{2}{6} + \frac{2.5}{6.9} + \frac{2.5.8}{6.9.12} + \frac{2.5.8.11}{6.9.12.15} \text{ \&c.}$$

74. Also from $2 \times : 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c $= \sqrt[3]{2}$ in the last article, we find $\frac{1}{2}\sqrt[3]{2} =$

$$1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \text{ \&c} = \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

75. In this case also, namely, $c = b$, the equation $d = \sqrt[3]{(b-c)} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3.6b^2}$ &c, becomes $0 = \sqrt[3]{b} \times : 1 - \frac{1}{3} - \frac{2}{3.6} - \frac{2.5}{3.6.9}$ &c.

And hence, dividing by $\sqrt[3]{b}$, and adding, we have

$$1 = \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} + \frac{2.5.8}{3.6.9.12} \text{ \&c,}$$

the same as in the last article but one.

76. And by taking other values of b and c, or other relations between them, any number of infinite series may be assigned, whose sums will be given by the two equations $\sqrt[3]{(b \pm c)} = \sqrt[3]{b} \times : 1 \pm \frac{c}{3b} - \frac{2c^2}{3.6b^2} \pm \frac{2.5c^3}{3.6.9b^3}$ &c. And if b be very great in respect of c, the two first terms of the series will give the cube root true to many places of figures.

77. Hitherto is concerning one of the limits or extreme cases only, namely, when $c^2 = b^2$, or when the equation is $x^3 = q = 2b$. And it has been observed, that the first general series for the three roots converges, in all the cases of the equation $x^3 - px = q$, or $x^3 - 3ax = 2b$, in which a^3 is not greater than $2b^2$. But a^3 may be any real quantity, not greater than $2b^2$, and so it may be either less than, equal to, or greater than b^2.

78. When, in this equation, a^3 is less than b^2, then c^2 is positive, and less than b^2, and the first series gives the only real root; without any change in the signs of the terms.

And to this belongs all cases of the equation that can fall in between the 13th and 16th formulæ, in the general table in Art. 30.

79. If a^3 be $= b^2$, then $c = 0$, and the three first series give $2\sqrt[3]{b} = \sqrt[3]{4q}$ for the greatest root, and $-\sqrt[3]{b}$ for each of the less roots. The same as at the 13th form in the general table Art. 30.

80. When a^3 is greater than b^2, c^2 will be negative, and then, changing the signs of the odd powers of c^2, the three general series will give the three roots of the equation, which will always be all real. In this class are two cases, namely, when c^2 is less than b^2, and when they are equal, which is the limit; for when c^2 becomes greater than b^2, the series diverge.

81. Now when a^3 is between b^2 and $2b^2$, then c^2 is negative and less than b^2, and the general series give all the three real roots, by changing the sign of every other term.

82. And when $a^3 = 2b^2$, then $-c^2 = b^2$, and the three roots become thus:

$2\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18}$ &c, the first or greatest root,

and $\left\{\begin{array}{l} -\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c,} \\ \pm\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c,} \end{array}\right\}$

the two less roots.

83. The first of these three is the greatest root, because $\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c, is greater than $\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c, for $1 + \frac{2}{3.6}$ &c, is greater than 1, and $\sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c, $= \sqrt{\frac{1}{3}} \times : 1 - \frac{2.5}{6.9}$ &c, is less than 1. So that in general the first series gives the greatest of the three roots.

84. But it is evident, that this case agrees with the 10th form in the table Art. 30; in which the middle root r is

found to be $\sqrt[3]{\frac{4q}{2}} = \sqrt[3]{2q} = -\sqrt[3]{4b} = -2\sqrt[3]{\frac{1}{2}b}$, and the other two, or the greatest and least roots, are $-\frac{1}{2}r \times (1 \pm \sqrt{3}) = \sqrt[3]{\frac{1}{2}b} \times (1 \pm \sqrt{3})$.

85. Hence, by a comparison of these two different forms of the same roots, we find

$$\frac{\sqrt{3}+1}{2\sqrt[3]{2}} = 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c} = \text{A},$$

and $$\frac{\sqrt{3}-1}{2\sqrt[3]{2}} = \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \text{\&c} = \text{B}.$$

86. And by adding and subtracting these two, we find

$$\frac{\sqrt{3}}{\sqrt[3]{2}} = 1 + \frac{1}{3} + \frac{2}{3.6} - \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + + - - \text{\&c, and}$$

$$\frac{1}{\sqrt[3]{2}} = 1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - + + - - \text{\&c,}$$
$= \text{C}$.

87. Also, because $\frac{\sqrt{3}+1}{2\sqrt[3]{2}} \times \frac{\sqrt{3}-1}{2\sqrt[3]{2}}$ is $= \frac{1}{2\sqrt[3]{4}}$, which is $= \frac{1}{2} \times (\frac{1}{\sqrt[3]{2}})^2$; therefore the mean proportional between the two series A and B, is to the series C, as the side of a square is to its diagonal.

88. Further, to and from the two series A and B, adding and subtracting the two series in Art. 74, namely, $\frac{1}{2}\sqrt[3]{2}$ or

$$\frac{\sqrt[3]{2}}{2} = 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \text{\&c} = \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c,}$$

we obtain the four following series:

$$\frac{\sqrt{3}+\sqrt[3]{4}+1}{4\sqrt[3]{2}} = 1 - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11.14.17.20}{3.6.9.12.15.18.21.24} \text{ \&c,}$$

$$\frac{\sqrt{3}+\sqrt[3]{4}-1}{4\sqrt[3]{2}} = \frac{1}{3} + \frac{2.5.8.11}{3.6.9.12.15} + \frac{2.5.8.11.14.17.20.23}{3.6.9.12.15.18.21.24.27} \text{\&c,}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}+1}{4\sqrt[3]{2}} = \frac{2}{3.6} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c,}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}-1}{4\sqrt[3]{2}} = -\frac{2.5}{3.6.9} - \frac{2.5.8.11.14.17}{3.6.9.12.15.18.21} \text{ \&c.}$$

89. It also appears, that the series

$$1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c,}$$

is the reciprocal of the series

$$1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c,}$$

where the signs of the former series are found by changing the signs of every other pair of terms in the latter; namely, omitting the first term, change the signs of the 2d and 3d terms, then passing over the 4th and 5th terms, change the signs of the 6th and 7th; and so on. For, by Art. 86, the former of these series is equal to $\frac{1}{\sqrt[3]{2}}$; and, by Art. 72, the latter is equal to $\sqrt[3]{2}$.

90. Let us now consider the cases in which c^2 is greater than b^2, which include all the cases not comprehended by the former, or in which c^2 is not greater than b^2. And this, it is evident, will happen both when a is positive, and when negative; namely when a is any positive quantity whatever, or when it is any negative quantity, and a^3 greater than $2b^2$. And in these two classes, c^2 will be positive or negative, according as a is positive or negative.

91. Now the series in this class will be found the same way as in the last, by only writing here the letter c before the letter b; for then we shall have $s = \sqrt[3]{(c+b)}$, and $d = \sqrt[3]{(-c+b)} = -\sqrt[3]{(c-b)}$.

Then $s = \sqrt[3]{(c+b)} = \sqrt[3]{c} \times : 1 + \frac{b}{3c} - \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3}$ &c,

and $d = -\sqrt[3]{(c-b)} = \sqrt[3]{c} \times : -1 + \frac{b}{3c} + \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3}$ &c.

Hence $s + d = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4}$ &c $=$ the 1st root, and which was given by Clairaut. And

$$\left.\begin{matrix} -\frac{s+d}{2} \\ \pm\frac{s-d}{2}\sqrt{-3} \end{matrix}\right\} = \left\{\begin{matrix} \frac{-b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \text{ \&c} \\ \pm\sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} - \frac{2 \cdot 5 \cdot 8b^4}{3 \cdot 6 \cdot 9 \cdot 12c^4} \text{ \&c}, \end{matrix}\right.$$

for the other two roots, which are new.

92. Here it again appears, that when c^2 is positive, the two latter roots are imaginary; because then $\sqrt[3]{c} \times \sqrt{-3}$ will be imaginary. But if c^2 be negative, those roots will be both real; since $\sqrt[3]{c} \times \sqrt{-3}$ then becomes $\sqrt[3]{(c\sqrt{-1})} \times \sqrt{-3} = \sqrt[3]{c} \times -\sqrt{-1} \times \sqrt{-3} = -\sqrt[3]{c} \times \sqrt{3}$. The signs prefixed to the terms as above, take place when c^2 is positive; but when c^2 shall be negative, the signs of the terms con-

taining the odd powers of it must be changed. And these series include all the cases in which the former ones failed, by not converging. So that, between them, they comprehend all the cases of the general cubic equation $x^3 \pm px = q$, as they each reciprocally converge when the other diverges, but in no other case, except in the common class, in which c is $= b$, which happens at the two limits, namely, either when a is $= 0$, or when $-a^3 = 2b^2$: and then they both give the same roots. But in the other cases they give the contrary roots; namely, when c is less than b, the first series gives the greatest root; and when c is greater than b, the latter series gives the least root.

93. Now when a is any positive quantity, the first of these series gives the only real root, without any change in the signs of the terms; the other two being imaginary. And this includes all the cases after the 16th in the table in Art. 30.

94. When a is $= 0$, or the limit between positive and negative, as in the 16th form in Art. 30, then is $c = b$, and the only real root, or the first series, becomes $2\sqrt[3]{b} \times : \frac{1}{3} + \frac{2.5}{3.6.9} +$ &c, which is the same root as was before found in Art. 73. So that, in this 16th case, both this series and the series in Art. 67, converge, and give the same and only real root.

95. When a becomes negative, then c^2 becomes negative, and the roots all real. But in this case the series only begins to converge when $-a^3 = 2b^2$, for then $-c^2$ becomes $= b^2$, and then, making the proper change in the signs of the terms, the three roots become

1st. $-2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15}$ &c, the least root, and

$$\begin{cases} +\sqrt[3]{b} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c,} \\ \pm\sqrt[3]{b} . \sqrt{3} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c,} \end{cases}$$

the two greater roots.

96. It has been here said, that the first of these three roots, is the least of them. To prove which, we assert, that $\sqrt{3}\times:1+\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}$ &c, is greater than 3 times $\frac{1}{3}-\frac{2.5}{3.6.9}$ &c; for $3\times:\frac{1}{3}-\frac{2.5}{3.6.9}$ &c $=1-\frac{2.5}{6.9}$ &c, is less than 1, whereas $1+\frac{2}{3.6}$ &c is greater than 1. Consequently, the less of the two latter roots, namely, $\sqrt[3]{b}\,.\,\sqrt{3}\times:1+\frac{2}{3.6}$ &c $-\sqrt[3]{b}\times:\frac{1}{3}-\frac{2.5}{3.6.9}$ &c, is greater than the first root $2\sqrt[3]{b}\times:\frac{1}{3}-\frac{2.5}{3.6.9}$ &c. That is to say, here the first is the least of the three roots, while in the other class of series the first is the greatest root.

97. Hence, comparing the value of any one of the roots here found, with the value of the same root as found in Art. 82, we obtain the relation between the two series that are concerned in them, namely, that the series $1+\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}+\frac{2.5.8.11.14}{3.6.9.12.15.18}$ &c, is to the series $\frac{1}{3}-\frac{2.5}{3.6.9}+\frac{2.5.8.11}{3.6.9.12.15}-\frac{2.5.8.11.14.17}{3.6.9.12.15.18.21}$ &c, as $\sqrt{3}+1$ is to $\sqrt{3}-1$, or as $2+\sqrt{3}$ to 1, or as 1 to $2-\sqrt{3}$, which are all equal to the same ratio. And the same thing appears from Art. 85.

98. When $-a^3$ becomes greater than $2b^2$, then $-c^3$ is greater than b^2; and, by the proper change in the signs, the series for the roots, in all cases of this kind, become

1st. $\frac{-2b}{\sqrt[3]{c^2}}\times:\frac{1}{3}-\frac{2.5b^2}{3.6.9c^2}+\frac{2.5.8.11b^4}{3.6.9.12.15c^4}$ &c, the least root;

and $\left\{\begin{array}{l}+\frac{b}{\sqrt[3]{c^2}}\times:\frac{1}{3}-\frac{2.5b^2}{3.6.9c^2}+\frac{2.5.8.11b^4}{3.6.9.12.15c^4}\text{ \&c,}\\ \pm\sqrt[3]{c}\,.\,\sqrt{3}\times:1+\frac{2b^2}{3.6c^2}-\frac{2.5.8b^4}{3.6.9.12c^4}\text{ \&c,}\end{array}\right\}$ the two greater roots.

99. Let us now illustrate all the foregoing series, for the roots of cubic equations, by finding, by means of them, the roots of the equations already treated of in Art. 56, &c.

100. And first, in the equation $x^3-36x=91$. Here

$p = -36$, $q = 91$, $a = -12$, $b = 45\frac{1}{2}$, $c^2 = b^2 + a^3 = (45\frac{1}{2})^2 - 12^3 = 342\frac{1}{4}$, which being positive, and less than b^2, this case belongs to the series

$2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} - \&c$, in Art. 68.

Now $\frac{c^2}{b^2} = \frac{1369}{8281} = (\frac{37}{91})^2 = \cdot 1653182$. Then

$$\begin{array}{llr}
\text{A} = & & + 1{\cdot}0000000 \\
\text{B} = \frac{2c^2}{3 \cdot 6b^2}\text{A} & = - & \cdot 0183687 \\
\text{C} = \frac{5 \cdot 8c^2}{9 \cdot 12b^2}\text{B} & = - & 11247 \\
\text{D} = \frac{11 \cdot 14c^2}{15 \cdot 18b^2}\text{C} & = - & 1061 \\
\text{E} = \frac{17 \cdot 20c^2}{21 \cdot 24b^2}\text{D} & = - & 118 \\
\text{F} = \frac{23 \cdot 26c^2}{27 \cdot 30b^2}\text{E} & = - & 14 \\
\text{G} = \frac{29 \cdot 32c^2}{33 \cdot 36b^2}\text{F} & = - & 2 \\
\hline
\text{sum of the terms} = & & \cdot 9803871 \\
 & & 2 \\
\hline
 & & 1{\cdot}9607742
\end{array}$$

$1{\cdot}9607742$ - log. $0{\cdot}2924275$

$\sqrt[3]{b} = \sqrt[3]{45{\cdot}5}$ - - - $0{\cdot}5526705$

hence the only real root is 7 - - - $0{\cdot}8450980$

That is, $x = 7$ is $= 2\sqrt[3]{\frac{91}{2}} \times : 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \&c.$

101. The other two roots are imaginary, and in Art. 56 they were found to be $= \frac{-7 \pm \sqrt{-3}}{2}$; but, by means of the series in Art. 68, they are here found to be

$\frac{-7}{2} \pm \frac{c\sqrt{-3}}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \&c.$ Consequently we obtain these following sums:

$$\frac{7}{2}\sqrt[3]{\frac{2}{91}} = 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 37^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 91^6} \&c,$$

$$\frac{1}{37}\sqrt[3]{\frac{91^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 37^2}{3 \cdot 6 \cdot 9 \cdot 91^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 91^4} \&c.$$

102. Ex. 2. In the equation $x^3+30x=117$, we have $a=\frac{1}{3}p=10$, $b=\frac{1}{2}q=\frac{117}{2}=58\frac{1}{2}$, and $c^2=b^2+a^3=(\frac{133}{2})^2$; which being positive, and greater than b^2, the proper series for this is that in Art. 91, namely,

$$x=\frac{2b}{\sqrt[3]{c^2}}\times:\frac{1}{3}+\frac{2.5b^2}{3.6.9c^2}+\frac{2.5.8.11b^4}{3.6.9.12.15c^4}+\&c.$$

Now $\frac{b^2}{c^2}=(\frac{117}{133})^2=\cdot7738308$. Hence

$A=\frac{1}{3}$	$=$	$\cdot333$		
$B=\frac{2.5b^2}{6.9c^2}A$	$=$	43	$\frac{2b}{\sqrt[3]{c^2}}$	$=7\cdot128$
$C=\frac{8.11b^2}{12.15c^2}B$	$=$	18	series inverted	124
$D=\frac{14.17b^2}{18.21c^2}C$	$=$	9		2851
				142
$E=\frac{20.23b^2}{24.27c^2}D$	$=$	5		7
$F=\frac{26.29b^2}{30.33c^2}E$	$=$	3	the root x	$=3\cdot000$
$G=\frac{32.35b^2}{36.39c^2}F$	$=$	2		
$H=\frac{38.41b^2}{42.45c^2}G$	$=$	1		
$I=\frac{44.47b^2}{48.51c^2}H$	$=$	1		
$K=\frac{50.53b^2}{54.57c^2}I$	$=$	1		
sum of the terms	$=$	$\cdot421$		

That is, $x=3=\frac{2.117}{\sqrt[3]{2.133^2}}\times:\frac{1}{3}+\frac{2.5.117^2}{3.6.9.133^2}+\&c.$

103. By the other series, in the same article, the two imaginary roots come out

$=-\frac{2}{3}\pm\sqrt[3]{c}.\sqrt{-3}\times:1-\frac{2b^2}{3.6c^2}-\&c$, which were before found, in Art. 57, to be $-\frac{3}{2}\pm\frac{7}{2}\sqrt{-3}$. Consequently

$$\frac{7}{2}\sqrt[3]{\frac{2}{133}}=1-\frac{2.117^2}{3.6.133^2}-\frac{2.5.8.117^4}{3.6.9.12.133^4}-\frac{2.5.8.11.14.117^6}{3.6.9.12.15.18.133^6}\&c;$$

$$\frac{1}{39}\sqrt[3]{\frac{133^2}{2^2}}=\frac{1}{3}+\frac{2.5.117^2}{3.6.9.133^2}+\frac{2.5.8.11.117^4}{3.6.9.12.15.133^4}+\&c.$$

104. Ex. 3. In the equation $x^3+18x=6$, we have $a=6$, $b=3$, $c=\sqrt{(9+216)}=\sqrt{225}=15$, real, and greater than b, and therefore this case belongs to the same series as the last example. Now $\frac{b^2}{c^2}=\frac{9}{225}=\frac{1}{25}=\cdot 04$, and $\frac{2b}{\sqrt[3]{c^2}}=\frac{6}{\sqrt[3]{225}}=\sqrt[3]{\frac{24}{25}}=\frac{1}{5}\sqrt[3]{120}=\sqrt[3]{\cdot 96}$. Then

$$\begin{aligned}
\text{A} &= \frac{1}{3} &&= \cdot 3333333 \\
\text{B} &= \frac{2 \,.\, 5b^2}{6 \,.\, 9c^2}\text{A} &&= \quad 24692 \\
\text{C} &= \frac{8 \,.\, 11b^2}{12 \,.\, 15c^2}\text{B} &&= \qquad 483 \\
\text{D} &= \frac{14 \,.\, 17b^2}{18 \,.\, 21c^2}\text{C} &&= \qquad 12
\end{aligned}$$

$$\begin{array}{lr}
\cdot 3358520 \text{ - - - - log.} & \bar{1}\cdot 5261480 \\
\sqrt[3]{\cdot 96} \text{ - - - - - - - -} & \bar{1}\cdot 9940904 \\
\text{the root } x = \cdot 3313130 \text{ - - - - - -} & \bar{1}\cdot 5202384
\end{array}$$

And then the two imaginary roots are

$$-\frac{\cdot 331313}{2} \pm \sqrt[3]{c} \,.\, \sqrt{-3} \times : 1 - \frac{2b^2}{3 \,.\, 6c^2} \text{ \&c.}$$

105. But, in Art. 58, these three roots were found to be $\sqrt[3]{18}-\sqrt[3]{12}$, and $-\frac{\sqrt[3]{18}-\sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18}+\sqrt[3]{12}}{2}\sqrt{-3}$. Consequently we have

$$\frac{\sqrt[3]{18}+\sqrt[3]{12}}{2\sqrt[3]{15}} = 1 - \frac{2}{3 \,.\, 6 \,.\, 25^2} - \frac{2 \,.\, 5 \,.\, 8}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 25^4} - \text{\&c, and}$$

$$\frac{\sqrt[3]{18}-\sqrt[3]{12}}{2}\sqrt[3]{\frac{25}{3}} = \frac{1}{3} + \frac{2 \,.\, 5}{3 \,.\, 6 \,.\, 9 \,.\, 25^2} + \frac{2 \,.\, 5 \,.\, 8 \,.\, 11}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15 \,.\, 25^4} + \text{\&c.}$$

106. Ex. 4. In the equation $x^3-15x=4$, we have $a=-5$, $b=2$, and $c=\sqrt{(b^2+a^3)}=\sqrt{-121}=11\sqrt{-1}$, imaginary, and greater than b, which belongs to the same series as the last two examples, but changing the sign where the odd powers of the negative quantity c^2 is concerned, as in Art. 98.

Now $\frac{b^2}{c^2}=\frac{2^2}{11^2}=\frac{4}{121}$, and $\frac{2b}{\sqrt[3]{c^2}}=\frac{4}{\sqrt[3]{121}}=\sqrt[3]{\frac{64}{121}}$. Then

$$\begin{array}{llr|llr}
 & & + & & & - \\
A = \frac{1}{3} & = & \cdot 3333333 & B = \frac{2 \,.\, 5b^2}{6 \,.\, 9c^2}A & = & \cdot 0020406 \\
C = \frac{8 \,.\, 11b^2}{12 \,.\, 15c^2}B & = & 330 & D = \frac{14 \,.\, 17b^2}{18 \,.\, 21c^2}C & = & 7 \\
\hline
 & + & \cdot 3333663 & & - & \cdot 0020413 \\
 & - & \cdot 0020413 & & & \\
\end{array}$$

$$\begin{array}{lrl}
\text{the series} = & \cdot 3313250 \text{ - - -} & \log.\ \bar{1}\cdot 5202543 \\
 & \sqrt[3]{\frac{64}{121}} \text{ - - - - - -} & \bar{2}\cdot 9077982 \\
\hline
\text{the least root} = - & \cdot 2679492 \text{ - - - - - -} & \bar{1}\cdot 4280525
\end{array}$$

107. To find the other roots by this method, we must sum the series $\sqrt[3]{c} \,.\, \sqrt{3} \times : 1 + \frac{2b^2}{3 \,.\, 6c^2} -$ &c. And as the terms of it are found by multiplying the terms A, B, C, &c, of the former, by $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c, respectively, we shall therefore have

$$\begin{array}{llr|llr}
\alpha = \frac{3}{1}A & = & 1 & \gamma = \frac{15}{11}C & = - & \cdot 0000450 \\
\beta = \frac{9}{5}B & = & 0\cdot 0036731 & & & \\
\delta = \frac{21}{17}D & = & 8 & & & \\
\hline
 & + & 1\cdot 0036739 & & & \\
 & - & 0000450 & & & \\
\end{array}$$

$$\begin{array}{lrl}
\text{series} = + & 1\cdot 0036289 \text{ - - - - - -} & \log.\ 0\cdot 0015732 \\
 & \sqrt[3]{11} \text{ - - - - - - - -} & 0\cdot 3471309 \\
 & \sqrt{3} \text{ - - - - - - - -} & 0\cdot 2385606 \\
\hline
 & \pm 3\cdot 8660254 \text{ - - - - - -} & 0\cdot 5872647
\end{array}$$

$$\left.\begin{array}{l}\tfrac{1}{2} \text{ the least root} \\ \text{with a contr. sign}\end{array}\right\} + 0\cdot 1339746$$

$$\begin{array}{lll}
\text{sum} & +4\cdot 0000000 & \text{greatest root} \\
\text{diff.} & -3\cdot 7320508 & \text{middle root.}
\end{array}$$

108. But the same 3 roots, found in Art. 59, are also 4, and $-2 \pm \sqrt{3}$; which being compared with the series in this example, we find

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2\,.\,2^2}{3\,.\,6\,.\,11^2} - \frac{2\,.\,5\,.\,8\,.\,2^4}{3\,.\,6\,.\,9\,.\,12\,.\,11^4} + \frac{2\,.\,5\,.\,8\,.\,11\,.\,14\,.\,2^6}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,18\,.\,11^6} \text{ \&c,}$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2\,.\,5\,.\,2^2}{3\,.\,6\,.\,9\,.\,11^2} + \frac{2\,.\,5\,.\,8\,.\,11\,.\,2^4}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,11^4} - \text{ \&c.}$$

109. Ex. 5. In the equation $x^3 - 6x = 4$, we have $a = -2$, $b = 2$, and $c^2 = b^2 + a^3 = 4 - 8 = -4$, which being negative, and $= b^2$, this case belongs to the series either in Art. 82 or 95. The operation of summing the terms, by them, is here omitted, because so much room would be necessary to set down so great a number of terms, and as the properties arising from the series in this case, have already been noticed above. The three roots of this equation have been found, in Art. 60, to be -2 and $1 \pm \sqrt{3}$.

110. Ex. 6. In the equation $x^3 - 9x = -10$, we have $a = -3$, $b = -5$, and $c^2 = 25 - 27 = -2$, which being negative, and less than b^2, the general series in Art. 68, with the necessary change of the signs, will give the three roots. Now $\frac{c^2}{b^2} = \frac{2}{25} = \frac{8}{100} = \cdot 08$, and $\sqrt[3]{b} = -\sqrt[3]{5}$, also $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}} = \frac{\sqrt{6}}{\sqrt[3]{25}}$. Hence

$$\begin{array}{llr|llr}
A = & & = 1 & & & \\
B = & \frac{2c^2}{3\,.\,6b^2}\,A & = 0\cdot 0088889 & C = & \frac{5\,.\,8c^2}{9\,.\,12b^2}\,B & = \cdot 0002634 \\
D = & \frac{11\,.\,14c^2}{15\,.\,18b^2}\,C & = \quad 120 & E = & \frac{17\,.\,20c^2}{21\,.\,24b^2}\,D & = \quad 7 \\
\hline
 & & + 1\cdot 0089009 & & & - \cdot 0002641 \\
 & & - 0\cdot 0002641 & & & \\
 & & + 1\cdot 0086368 & & & \\
 & & 2 & & &
\end{array}$$

$$\begin{array}{lr}
2\cdot 0172736 \quad \text{- - - -} \quad \log. & 0\cdot 3047649 \\
\sqrt[3]{b} = \sqrt[3]{5} \quad \text{- - - - - - -} & 0\cdot 2329900 \\
\hline
\text{the greatest root} = -3\cdot 44948974 \quad \text{- - - -} & 0\cdot 5377549
\end{array}$$

111. Then, for the other roots, by multiplying the terms A, B, C, &c, of the former, by $\frac{1}{3}$, $\frac{5}{9}$, $\frac{11}{15}$, &c, we have

$\alpha = \frac{1}{3}$ A	$= \cdot 3333333$		
$\gamma = \frac{11}{15}$ C	$= \quad 1931$	$\beta = \frac{5}{9}$ B	$= \cdot 0049383$
$\varepsilon = \frac{23}{27}$ E	$= \quad 6$	$\delta = \frac{17}{21}$ D	$= \quad 97$
	$+ \cdot 3335270$		$- \cdot 0049480$
	$- \cdot 0049480$		

$\cdot 3285790$ - - - - log. $\bar{1}\cdot 5166398$

$\frac{\sqrt{6}}{\sqrt[3]{25}}$ - - - - - - - - - $\bar{1}\cdot 9230956$

the second series $\pm$ $\cdot 27525513$ - - - $\bar{1}\cdot 4397354$

$\frac{1}{2}$ the greatest root $+$ $1\cdot 72474487$

middle root $2\cdot 00000000$

least root $1\cdot 44948974$

112. But, by Art. 61, these three roots were found to be 2 and -1 ± 6; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{6}+1}{2\sqrt[3]{5}} = 1 + \frac{2 \cdot 2}{3 \cdot 6 \cdot 25} - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^3}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 25^3} \&c,$$

$$\frac{\sqrt{6}-2}{4}\sqrt[3]{25} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2}{3 \cdot 6 \cdot 9 \cdot 25} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^2} - \&c.$$

113. Ex. 7. In the equation $x^3 - 12x = 9$, we have $a = -4$, $b = \frac{9}{2}$, and $c^2 = \frac{81}{4} - 64 = -\frac{175}{4}$, which being negative, and greater than b^2, we shall have 3 real roots by the series in Art. 98.

Now $\frac{b^2}{c^2} = \frac{81}{175}$, $\frac{b}{\sqrt[3]{c^2}} = \frac{9}{\sqrt[3]{350}} = \sqrt[3]{\frac{729}{350}}$, and

$\sqrt[3]{c} = \sqrt[6]{\frac{175}{4}} = \sqrt[6]{43\cdot 75}$. Then

$A = \frac{1}{3}$	$= \cdot33333$	$B = \frac{2.5b^2}{6.9c^2}A$	$= \cdot02857$
$C = \frac{8.11b^2}{12.15c^2}B$	$= 647$	$D = \frac{14.17b^2}{18.21c^2}C$	$= 188$
$E = \frac{20.23b^2}{24.27c^2}D$	$= 62$	$F = \frac{26.29b^2}{30.33c^2}E$	$= 22$
$G = \frac{32.35b^2}{36.39c^2}F$	$= 8$	$H = \frac{38.41b^2}{42.45c^2}G$	$= 3$
$I = \frac{44.47b^2}{48.51c^2}H$	$= 1$	$K = \frac{50.53b^2}{54.57c^2}I$	$= 1$
	$+ \cdot34051$		$- \cdot03071$

$$
\begin{array}{rl}
+ & \cdot34051 \\
- & \cdot03071 \\
\hline
 & \cdot30980 \\
 & 2 \\
\hline
\end{array}
$$

	$\cdot61960$ - - - - log.	$\bar{1}\cdot7921114$
	$\sqrt[3]{\frac{729}{350}}$ - - - - - - - -	$0\cdot1062198$
the least root $= -$	$\cdot79128$ - - - - - - -	$\bar{1}\cdot8983312$

114. Then, since the terms of the latter series are found by multiplying the terms of the former by the fractions $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c, they will be thus:

$\alpha = \frac{3}{1}A$	$= 1\cdot00000$		
$\beta = \frac{9}{5}B$	$= 5143$	$\gamma = \frac{15}{11}C$	$= \cdot00882$
$\delta = \frac{21}{17}D$	$= 232$	$\epsilon = \frac{27}{23}E$	$= 73$
$\zeta = \frac{33}{29}F$	$= 25$	$\eta = \frac{39}{35}G$	$= 9$
$\theta = \frac{45}{41}H$	$= 4$	$\iota = \frac{51}{47}I$	$= 1$
	$+ 1\cdot05404$		$- \cdot00965$

	$+ 1\cdot05404$	
	$- 0\cdot00965$	
	$1\cdot04439$ - - - - log.	$0\cdot0188627$
	$\sqrt[6]{43\cdot75}$ - - - - - -	$0\cdot2734963$
	$\sqrt{3}$ - - - - - - -	$0\cdot2385606$
last series $\pm$	$3\cdot39564$ - - - - - -	$0\cdot5309196$
$-\frac{1}{2}$ the first $+$	$0\cdot39564$	
greatest root $+$	$3\cdot79128$	
middle root $-$	$3\cdot00000$	

115. But, by Art. 62, these same three roots are, -3, and $\frac{3\pm\sqrt{21}}{2}$; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{21}+9}{12\sqrt[6]{350}}\sqrt{6}=1+\frac{2.81}{3.6.175}-\frac{2.5.8.81^2}{3.6.9.12.175^2}+\&c, \text{ and}$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350}=\frac{1}{3}-\frac{2.5.81}{3.6.9.175}+\frac{2.5.8.11.81^2}{3.6.9.12.15.175^2}-\&c.$$

116. Ex. 8. In the equation $x^3-12x=-8\sqrt{2}$, we have $a=-4$, $b=-4\sqrt{2}$, and $c^2=32-64=-32$; which being negative, and equal to b^2, the three roots will be found, by both the forms of series, like as in Ex. 5, Art. 109; but the operation is here omitted for the same reasons as were there given. The three roots of this equation were, in Art. 63, found to be $2\sqrt{2}$ and $-\sqrt{2}\pm\sqrt{6}$.

117. Ex. 9. In the equation $x^3-15x=22$, we have $a=-5$, $b=11$, and $c^2=121-125=-4$; which being negative, and less than b^2, the series in Art. 68 give these three roots:

Greatest root $=2\sqrt[3]{11}\times:1+\frac{2c^2}{3.6b^2}-\frac{2.5.8c^4}{3.6.9.12b^4}\ \&c$, and

the two less roots $\left\{\begin{array}{l}-\sqrt[3]{11}\times:1+\frac{2c^2}{3.6b^2}-\frac{2.5.8c^4}{3.6.9.12b^4}\ \&c,\\ \pm\frac{2\sqrt{3}}{\sqrt[3]{121}}\times:\frac{1}{3}-\frac{2.5c^2}{3.6.9b^2}+\frac{2.5.8.11c^4}{3.6.9.12.15b^4}\ \&c;\end{array}\right\}$ where $\frac{c^2}{b^2}=\frac{4}{121}$.

Here

$$\begin{array}{lll|l}
\text{A} = & & = 1{\cdot}0000000 & \\
\text{B} = \frac{2c^2}{3.6b^2}\text{A} & = & 36731 & \text{C} = \frac{5.8c^2}{9.12b^2}\text{B} = -{\cdot}0000450 \\
\text{D} = \frac{11.14c^2}{15.18b^2}\text{C} & = & 8 & \\
\hline
 & & +\ 1{\cdot}0036739 & \\
 & & -\ 0{\cdot}0000450 & \\
\hline
 & & 1{\cdot}0036289 & \\
 & & 2 & \\
\hline
 & & 2{\cdot}0072578 & \text{- - - - log. } 0{\cdot}3026031 \\
 & & \sqrt[3]{11} & \text{- - - - - - - } 0{\cdot}3471309 \\
\hline
\text{the greatest root} & = & 4{\cdot}4641016 & \text{- - - - - - - } 0{\cdot}6497340
\end{array}$$

118. Again,

$$\begin{array}{lcr|lcr}
\alpha = \frac{1}{3}\,A & = & \cdot3333333 & \beta = \frac{5}{9}\,B & = & \cdot0020406 \\
\gamma = \frac{11}{15}\,C & = & 330 & \delta = \frac{17}{21}\,D & = & 7 \\
\hline
 & & +\ \cdot3333663 & & & -\ \cdot0020413 \\
 & & -\ \cdot0020413 & & & \\
\hline
 & & \cdot3313250 & & & \\
 & & 2 & & & \\
\hline
\end{array}$$

$$\begin{array}{lrr}
 & \cdot6626500 & \text{log. } \bar{1}\cdot8212842 \\
 & \sqrt{3} & 0\cdot2385606 \\
 & \sqrt[3]{121} & 0\cdot6942618 \\
\hline
\text{the latter series} = & \cdot2320508 & \bar{1}\cdot3655830 \\
\frac{1}{2} \text{ the first} = & 2\cdot2320508 & \\
\hline
\text{middle root} = & -\ 2\cdot4641016 & \\
\text{least root} = & -\ 2\cdot0000000 &
\end{array}$$

119. But, by Art. 64, the three roots are -2 and $1 \pm \sqrt{12}$; hence

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \,.\, 2^2}{3 \,.\, 6 \,.\, 11^2} - \frac{2 \,.\, 5 \,.\, 8 \,.\, 2^4}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 11^4} + \frac{2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \,.\, 2^6}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15 \,.\, 18 \,.\, 11^6} \text{ \&c,}$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2 \,.\, 5 \,.\, 2^2}{3 \,.\, 6 \,.\, 9 \,.\, 11^2} + \frac{2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 2^4}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15 \,.\, 11^4} - \text{\&c.}$$

120. And in this manner the roots of cubic equations may always be found by these series; and then, by comparing them with the roots of the same equations, as found by other methods, we shall obtain as many series as we please, whose sums will be given.

121. Hence also we may find the sum of any general series of either of these forms, namely,

$$1 \mp \frac{2g^2}{3 \,.\, 6} - \frac{2 \,.\, 5 \,.\, 8g^4}{3 \,.\, 6 \,.\, 9 \,.\, 12} \pm \frac{2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14g^6}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15 \,.\, 18} \text{ \&c, or}$$

$$\frac{1}{3} \pm \frac{2 \,.\, 5g^2}{3 \,.\, 6 \,.\, 9} + \frac{2 \,.\, 5 \,.\, 8 \,.\, 11g^4}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15} \pm \frac{2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \,.\, 17g^6}{3 \,.\, 6 \,.\, 9 \,.\, 12 \,.\, 15 \,.\, 18 \,.\, 21} \text{ \&c, by}$$

comparing them with the roots of given cubic equations; whatever be the value of g, not greater than 1.

122. For, by Art. 68, $\sqrt[3]{(b+c)} + \sqrt[3]{(b-c)} = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \,.\, 6b^2} - \frac{2 \,.\, 5 \,.\, 8c^4}{3 \,.\, 6 \,.\, 9 \,.\, 12b^4}$ &c, is $=$ the greatest root of the cubic

equation $x^3 - 3\sqrt[3]{(b^2-c^2)}\,.\,x = 2b$. Now make $2\sqrt[3]{b} = 1$, and $\frac{c^2}{b^2} = g^2$; so shall the above become $\frac{1}{2}\sqrt[3]{(1+g)} + \frac{1}{2}\sqrt[3]{(1-g)}$ $= 1 - \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12}$ &c = the greatest root of the equation $x^3 - \frac{3}{4}\sqrt[3]{(1-g^2)}\,.\,x = \frac{1}{4}$. And when g^2 or $\frac{c^2}{b^2}$ is negative, these become

$\frac{1}{2}\sqrt[3]{(1+g\sqrt{-1})} + \frac{1}{2}\sqrt[3]{(1-g\sqrt{-1})} = 1 + \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} +$ &c = the greatest root of the equation $x^3 - \frac{3}{4}\sqrt[3]{(1+g^2)}x = \frac{1}{4}$. So that, in general, the infinite series

$1 \mp \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} \mp \frac{2.5.8.11.14g^6}{3.6.9.12.15.18}$ &c, is $=$ $\frac{1}{2}\sqrt[3]{(1+g\sqrt{\pm1})} + \frac{1}{2}\sqrt[3]{(1-g\sqrt{\pm1})}$ = the greatest root of the equation $x^3 - \frac{3}{4}\sqrt[3]{(1\mp g^2)}\,x = \frac{1}{4}$. Where the upper and under signs respectively correspond to each other.

123. Again,

$\sqrt[3]{(c+b)} - \sqrt[3]{(c-b)} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4}$ &c, is the least root of the equation $x^3 + 3\sqrt[3]{(c^2-b^2)}\,x = 2b$. Then, by taking $\frac{2b}{\sqrt[3]{c^2}} = 1$, and $\frac{b^2}{c^2} = g^2$, this becomes $\frac{\sqrt[3]{(1+g)}-\sqrt[3]{(1-g)}}{2g} = \frac{1}{3} + \frac{2.5g^2}{3.6.9}$ &c = the least root of the equation $x^3 + \frac{3\sqrt[3]{(1-g^2)}}{4g^2}\,x = \frac{1}{4g^2}$. And when g^2 or c^2 is negative, this becomes

$\frac{\sqrt[3]{(1+g\sqrt{-1})}-\sqrt[3]{(1-g\sqrt{-1})}}{2g\sqrt{-1}} = \frac{1}{3} - \frac{2.5g^2}{3.6.9} +$ &c = the least root of the equation $x^3 - \frac{3\sqrt[3]{(1+g^2)}}{4g^2}\,x = \frac{-1}{4g^2}$. So that, in general, the infinite series

$\frac{1}{3} \pm \frac{2.5g^2}{3.6.9} + \frac{2.5.8.11g^4}{3.6.9.12.15} \pm \frac{2.5.8.11.14.17g^6}{3.6.9.12.15.18.21}$ &c, is $= \frac{\sqrt[3]{(1+g\sqrt{\pm1})}-\sqrt[3]{(1-g\sqrt{\pm1})}}{2g\sqrt{\pm1}}$ = the least root of the equation $x^3 \pm \frac{3\sqrt[3]{(1\mp g^2)}}{4g^2}\,x = \frac{\pm1}{4g^2}$.

Of the Roots by another Class of Series.

124. But there are yet other series, converging much faster than those in the foregoing class, by the help of which, and Cardan's rule conjointly, may always be found the roots of those equations, in which that rule fails when it is applied singly, that is, in what is called the irreducible case, or that in which c^2 is negative. And these series are found by introducing another cubic equation, having the same values of b and c^2, as the given equation, except that, in the new equation, the value of c^2 is positive, while in the given one it is negative. For when c^2 is positive, the new equation to which it belongs, has only one real root, and that root is always found by Cardan's rule; but the contrary takes place when c^2 is negative, the equation having then three real roots, though they are not always determinable by that rule, because the radical quantities can seldom be extracted, on account of the square root of the negative quantity, which is contained in them.

125. Now the general expression for the root, by Cardan's rule, being $s + d = \sqrt[3]{(b + \sqrt{\pm c^2})} + \sqrt[3]{(b - \sqrt{\pm c^2})}$, or $\sqrt[3]{(\sqrt{\pm c^2} + b)} - \sqrt[3]{(\sqrt{\pm c^2} - b)}$, if the cubic roots of each of these be extracted, by the binomial theorem, as at Art. 68, we shall obtain these four forms:

1. $\sqrt[3]{(b + \sqrt{+c^2})} + \sqrt[3]{(b - \sqrt{+c^2})} = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \&c.$

2. $\sqrt[3]{(b + \sqrt{-c^2})} + \sqrt[3]{(b - \sqrt{-c^2})} = 2\sqrt[3]{b} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \&c.$

3. $\sqrt[3]{(\sqrt{+c^2} + b)} - \sqrt[3]{(\sqrt{+c^2} - b)} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \&c.$

4. $\sqrt[3]{(\sqrt{-c^2} + b)} - \sqrt[3]{(\sqrt{-c^2} - b)} = \frac{2b}{\sqrt[3]{c^2}} \times : -\frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} - \&c.$

126. Of which, the series in the first and third denote the only real root of the equation, when c^2 is positive, according as c is greater or less than b, which root call x; and the series in the second and fourth forms denote the greatest and least roots of the equation, when c^2 is negative, which

roots call R and r respectively. Then, by adding and subtracting the first and second, as also the third and fourth, there result these four equations:

$$R+X=4\sqrt[3]{b}\times:1-\frac{2.5.8c^4}{3.6.9.12b^4}-\frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8}\ \&c.$$

$$R-X=4\sqrt[3]{b}\times:\frac{2c^2}{3.6b^2}+\frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6}+\&c.$$

$$X-r=\frac{4b}{\sqrt[3]{c^2}}\times:\frac{1}{3}+\frac{2.5.8.11b^4}{3.6.9.12.15c^4}+\&c.$$

$$X+r=\frac{4b}{\sqrt[3]{c^2}}\times:\frac{2.5b^2}{3.6.9c^2}+\frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6}+\&c.$$

127. And hence, by equal addition or subtraction, we find these two different expressions, both for the greatest and least roots of a cubic equation, in which c^2 or b^2+a^3 is negative, namely,

$$R=-X+4\sqrt[3]{b}\times:1-\frac{2.5.8c^4}{3.6.9.12b^4}-\frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8}\ \&c,\text{ or}$$

$$R=X+4\sqrt[3]{b}\times:\frac{2c^2}{3.6b^2}+\frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6}+\&c;$$

$$r=X-\frac{4b}{\sqrt[3]{c^2}}\times:\frac{1}{3}+\frac{2.5.8.11b^4}{3.6.9.12.15c^4}+\frac{2.5.8.11.14.17.20.23b^8}{3.6.9.12.15.18.21.24.27c^8}\ \&c,\text{ or}$$

$$r=-X+\frac{4b}{\sqrt[3]{c^2}}\times:\frac{2.5b^2}{3.6.9c^2}+\frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6}+\&c;$$

where R is the greatest, and r the least root of the equation $x^3-3ax=2b$, or $x^3-3\sqrt[3]{(c^2+b^2)}\ x=2b$, and X the only real root of the equation $x^3+3\sqrt[3]{(c^2-b^2)}\ x=2b$; in which, as well as in the above series, c^2 denotes a positive quantity.

128. And hence it can no longer be said that Cardan's rule is of no use in the solution of cubic equations, that have three real roots; since they have here been reduced to the other case, in which the equation has only one real root; which case is always resolvable by that rule. And the first hint of such reduction I received from Francis Maseres, Esq. Cursitor Baron of the Exchequer, he having done me the favour to communicate to me the second of the above four forms for the greatest root, in a letter of the 17th of July 1779; the investigation of which formula, together with those of the other three, nearly as above, I had the

honour of sending to him, in a letter of the 26th of the same month; and that learned gentleman has since communicated to the Royal Society his said formula, together with his own investigation of it, done in his usual very accurate manner. Since that time I have seen, in the *Memoires de l'Acad.* for the year 1743, four expressions similar to the above, given by Mr. Nicole, for the purpose of summing certain terms of a binomial raised to any power, but unaccompanied with any appearance of the idea of thus reducing the one case of the cubic equation, to the other.

129. It is hardly necessary to remark, that any general series, of each of the above four forms, is summed by means of the sum or difference of the roots of these two equations, $x^3 - 3\sqrt[3]{(b^2 \pm c^2)}x = 2b$; and that, by substituting particular numbers for b and c, we may thus sum as many series of those forms as we please.

130. Ex. 1. We may now illustrate these formulas by some examples. And first, in the equation $x^3 - 15x = 4$. Here $2b = 4$, and $3\sqrt[3]{(b^2 + c^2)} = 15$, consequently $b = 2$, and $c^2 = 5^3 - b^2 = 125 - 4 = 121 = 11^2$, and $x = \sqrt[3]{(c + b)} - \sqrt[3]{(c - b)} = \sqrt[3]{13} - \sqrt[3]{9} = \cdot 2712508$, the root of the equation $x^3 - 3\sqrt[3]{(b^2 - c^2)}x = 2b$, or $x^3 + 3\sqrt[3]{117}\ x = 4$. And, as b is less than c, this equation belongs to the two series in the latter case for finding the least root. Hence, the terms of the two series agreeing with the positive and negative terms of the series in Art. 106, they will stand thus:

By the 1st series		By the 2d series	
A = ·3333333		B = ·0020406	
C = ·0000330		D = ·0000007	
·3333663 -	log. $\bar{1}$·5229217	·0020413 - -	log. $\bar{3}$·3099068
$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$ - - -	0·2088282	$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$ - - - -	0·2088282
series = − ·5392000 -	$\bar{1}$·7317499	series = + ·0033016 -	$\bar{3}$·5187350
x = + ·2712508 -		x = − ·2712508	
r = − ·2679492 the least root.		r = − ·2679492 the same root.	

Agreeing with the same root found in Ex. 4, Art. 106.

131. But the same root has been found to be $-2+\sqrt{3}$, in Art. 59; and hence we obtain the sums of these two particular series, thus:

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}+2-\sqrt{3}}{8}\sqrt[3]{121}=\frac{1}{3}+\frac{2\,.\,5\,.\,8\,.\,11\,.\,2^4}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,11^4}+\&c,$$

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}-2+\sqrt{3}}{8}\sqrt[3]{121}=\frac{2\,.\,5\,.\,2^2}{3\,.\,6\,.\,9\,.\,11^2}+\frac{2\,.\,5\,.\,8\,.\,11\,.\,14\,.\,17\,.\,2^6}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,18\,.\,21\,.\,11^6}\,\&c.$$

132. Also, by taking the sum and difference of these two, we have

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}}{4}\sqrt[3]{121}=\frac{1}{3}+\frac{2\,.\,5\,.\,2^2}{3\,.\,6\,.\,9\,.\,11^2}+\frac{2\,.\,5\,.\,8\,.\,11\,.\,2^4}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,11^4}+\&c, \text{ and}$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121}=\frac{1}{3}-\frac{2\,.\,5\,.\,2^2}{3\,.\,6\,.\,9\,.\,11^2}+\frac{2\,.\,5\,.\,8\,.\,11\,.\,2^4}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,11^4}-\&c.$$

And this last expression agrees with what was found in Art. 108.

133. Ex. 2. Again, in the equation $x^3-9x=-10$, we have $2b=-10$, and $3\sqrt[3]{(b^2+c^2)}=9$; consequently $b=-5$, and $c^2=3^3-b^2=27-25=2$; which being less than b^2 or 25, this equation belongs to the first class of series, or that for the greatest root. Now

$x=\sqrt[3]{(b+c)}+\sqrt[3]{(b-c)}=\sqrt[3]{(-5+\sqrt{2})}+\sqrt[3]{(-5-\sqrt{2})}$
$=-\sqrt[3]{(5-\sqrt{2})}-\sqrt[3]{(5+\sqrt{2})}$
$=-\sqrt[3]{3\cdot58578864}-\sqrt[3]{6\cdot41421356}$
$=-1\cdot530600-1\cdot858009=-3\cdot388609=$ the root of the equation $x^3-3\sqrt[3]{(b^2-c^2)}\,.\,x=2b$, or $x^3-3\sqrt[3]{(21)}\,.\,x=-10$. And the terms of the two series are found as in Art. 110, namely $1-\frac{2\,.\,5\,.\,8c^4}{3\,.\,6\,.\,9\,.\,12b^4}-\&c=\text{A}-\text{C}-\text{E}-\&c=\cdot9997359$, and $\frac{2c^2}{3\,.\,6b^2}+\&c=\text{B}+\text{D}+\&c=\cdot0089009$. Also $4\sqrt[3]{b}=4\sqrt[3]{-5}=-4\sqrt[3]{5}=-\sqrt[3]{320}$. Then

By the 1st series

	$\cdot9997359$	log. $\overline{1}\cdot9998854$
	$-\sqrt[3]{320}$ - - -	$0\cdot8350500$
series	$=-6\cdot838098$ -	$0\cdot8349354$
x	$=+3\cdot388609$	
	$-3\cdot449489$ the greatest root.	

By the 2d series

	$\cdot0089009$	log. $\overline{3}\cdot9494339$
	$-\sqrt[3]{320}$ - - -	$0\cdot8350500$
series	$=-\cdot060881$ - -	$\overline{2}\cdot7844839$
x	$=-3\cdot388609$	
	$-3\cdot449490$ the same root.	

And these values of the greatest root are nearly the same with that found in Art. 110.

134. But, in Art. 61, the same root was found to be $-1-\sqrt{6}$; hence we obtain the sums of these first two particular series; and by the addition and subtraction of these two, arise the other two following them, namely,

$$\frac{1+\sqrt{6}+\sqrt[3]{(5+\sqrt{2})}+\sqrt[3]{(5-\sqrt{2})}}{4\sqrt[3]{5}}=1-\frac{2.5.8.2^2}{3.6.9.12.5^4}-\&c;$$

$$\frac{1+\sqrt{6}-\sqrt[3]{(5+\sqrt{2})}-\sqrt[3]{(5-\sqrt{2})}}{4\sqrt[3]{5}}=\frac{2.2}{3.6.5^2}+\frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6}+\&c;$$

$$\frac{1+\sqrt{6}}{2\sqrt[3]{5}}=1+\frac{2.2}{3.6.5^2}-\frac{2.5.8.2^2}{3.6.9.12.5^4}+\frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6}-\&c;$$

$$\frac{\sqrt[3]{(5+\sqrt{2})}+\sqrt[3]{(5-\sqrt{2})}}{2\sqrt[3]{5}}=1-\frac{2.2}{3.6.5^2}-\frac{2.5.8.2^2}{3.6.9.12.5^4}-\&c.$$

And the last but one of these equations agrees with one found in Art. 112.

135. Ex. 3. Also, in the equation $x^3-12x=9$, we have $2b=9$, and $\sqrt[3]{(b^2+c^2)}=4$; consequently $b=\frac{9}{2}$, and $c^2=4^3-b^2=64-\frac{81}{4}=\frac{175}{4}$, which being greater than b^2 or $\frac{81}{4}$, this case belongs to the second class of series, or that of the least roots. Now here $x=\sqrt[3]{(c+b)}-\sqrt[3]{(c-b)}=\sqrt[3]{\frac{\sqrt{175}+9}{2}}-\sqrt[3]{\frac{\sqrt{175}-9}{2}}=\sqrt[3]{11{\cdot}114378}-\sqrt[3]{2{\cdot}114378}=2{\cdot}2316619-1{\cdot}2834950={\cdot}9481669=$ the root of the equation $x^3-3\sqrt[3]{(b^2-c^2)}\,.\,x=2b$, or $x^3+3\sqrt[3]{\frac{47}{2}}\,.\,x=9$. And the terms of the two series being found as in Art. 113, namely, A + C + E + &c $={\cdot}34051$, and B + D + F + &c $={\cdot}03071$, also $\frac{4b}{\sqrt[3]{c^2}}$ being $=\frac{36}{\sqrt[3]{250}}$, we shall have

By the 1st series			By the latter series		
	·34051	log. $\bar{1}{\cdot}5321299$		·03071	log. $\bar{2}{\cdot}4872798$
	$\frac{36}{\sqrt[3]{350}}$	0·7082798		$\frac{36}{\sqrt[3]{350}}$	0·7082798
series	= − 1·739441	0·2404097	series	= + ·1568771	$\bar{1}{\cdot}1955596$
x	= + 0·948167		x	= − ·9481669	
	− ·791274 the least root.			− ·7912898 the same root.	

Which nearly agree with the same root found in Art. 113.

136. But, in Art. 62, the same root was found to be $\frac{3-\sqrt{21}}{2}$; hence then we shall have these first two following equations, and by means of their sum and difference we obtain the other two:

$$\frac{\sqrt[3]{(20\sqrt{7}+36)}-\sqrt[3]{(20\sqrt{7}-36)}+\sqrt{21}-3}{72}\sqrt[3]{350}=\frac{1}{3}+\frac{2.5.8.11.81^2}{3.6.9,12.15.175^2}\ \&c\,;$$

$$\frac{\sqrt[3]{(20\sqrt{7}+36)}-\sqrt[3]{(20\sqrt{7}-36)}-\sqrt{21}+3}{72}\sqrt[3]{350}=\frac{2\,.\,5\,.\,81}{3\,.\,6\,.\,9\,.\,175}+\&c\,;$$

$$\frac{\sqrt[3]{(20\sqrt{7}+36)}-\sqrt[3]{(20\sqrt{7}-36)}}{36}\sqrt[3]{350}=\frac{1}{3}+\frac{2\,.\,5\,.\,81}{3\,.\,6\,.\,9\,.\,175}+\&c\,;$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350}=\frac{1}{3}-\frac{2\,.\,5\,.\,81}{3\,.\,6\,.\,9\,.\,175}+\frac{2\,.\,5\,.\,8\,.\,11\,.\,81^2}{3\,.\,6\,.\,9\,.\,12\,.\,15\,.\,175^2}-\&c.$$

And the last of these agrees with one found in Art. 115.

137. Ex. 4. In the equation $x^3-15x=22$, we have $2b=22$, and $\sqrt[3]{(b^2+c^2)}=5$; consequently $b=11$, and $c^2=5^3-b^2=125-121=4$; which being less than b^2, or 121, this belongs to the first class of series, or that for the greatest root.

Now $x=\sqrt[3]{(b+c)}+\sqrt[3]{(b-c)}=\sqrt[3]{13}+\sqrt[3]{9}=4{\cdot}4314186=$ the root of the equation $x^3-3\sqrt[3]{(117)}\,.\,x=22$. And the terms of the two series being found as in Art. 117, we have the first $=\text{A}-\text{C}-\&c=1-{\cdot}0000450=$ ${\cdot}9999550$, and the second $=\text{B}+\text{D}+\&c={\cdot}0036731+$ ${\cdot}0000008={\cdot}0036739$. Also $4\sqrt[3]{b}=4\sqrt[3]{11}=\sqrt[3]{704}$. Hence,

By the 1st series			By the 2d series		
	·9999550	log. $\bar{1}$·9999805		·0036739	log. $\bar{3}$·5651273
	$\sqrt[3]{704}$	0·9491909		$\sqrt[3]{704}$	0·9491909
series	= + 8·8955200	0·9491714	series	= + ·0326827	$\bar{2}$·5143182
x	= − 4·4314186		x	= + 4·4314186	
	+ 4·4641014 greatest root.			+ 4·4641013 the same root.	

Which nearly agree with the same root found in Art. 117.

138. But in Art. 64, the same root was found to be $1+\sqrt{12}=1+2\sqrt{3}$, hence we obtain these two first equations following, and their sum and difference give the other two:

$$\frac{1+\sqrt{12}+\sqrt[3]{13}+\sqrt[3]{9}}{-\sqrt[3]{11}}=1-\frac{2.5.8.2^4}{3.6.9.12.11^4}-\frac{2.5.8.11.14.17.20.2^8}{3.6.9.12.15.18.21.24.11^8}-\&c;$$

$$\frac{1+\sqrt{12}-\sqrt[3]{13}-\sqrt[3]{9}}{+\sqrt[3]{11}}=\frac{2.2^2}{3.6.11^2}+\frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6}+\&c;$$

$$\frac{\sqrt[3]{13}+\sqrt[3]{9}}{2\sqrt[3]{11}}=1-\frac{2.2^2}{3.6.11^2}-\frac{2.5.8.2^4}{3.6.9.12.11^4}-\frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6}--\&c;$$

$$\frac{1+\sqrt{12}}{2\sqrt[3]{11}}=1+\frac{2.2^2}{3.6.11^2}-\frac{2.5.8.2^4}{3.6.9.12.11^4}+\frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6}-+\&c.$$

The last of which agrees with one found in Art. 119.

And thus we may find the sums of as many series of these kinds as we please; as well as the sum of any of the general series, by means of the roots of given cubic equations. And thus also may series, by means of the roots of equations of other orders, be exhibited.

TRACT XXIX.

PROJECT FOR A NEW DIVISION OF THE QUADRANT. READ AT THE ROYAL SOCIETY, NOV. 27, 1783.

Having long since thought it would be a meritorious and useful service, to adapt the tables of sines, tangents, and secants, to equal parts of the radius, instead of to those of the quadrant; the following observations on the subject are thrown together, with a view to stimulate others, either to undertake and calculate some part of so large and painful a work, or to communicate further hints for the improvement and easier performance of it. This project then is for constructing sines, tangents, secants, &c, to equal parts of the radius.

1. The arbitrary division of the quadrant of the circle into equal parts, by 60ths, which has been delivered down to us from the ancients, and gradually extended by similar

sub-divisions by the moderns, among various uses, serves for trigonometrical and other mathematical operations, by adapting to those divisions of the arc, certain lines expressed in equal parts of the radius, as chords, sines, tangents, &c. But among all the improvements in this useful branch of science, I have long wished to see a set of tables of sines, tangents, secants, &c, constructed to the arcs of the quadrant as divided into the like equal parts of the radius as those lines themselves. In this natural way, the arcs would not be expressed by divisions of 60ths, in degrees, minutes, &c, but by the common decimal scale of numbers; and the real lengths of the arcs, expressed in such common numbers, would then stand opposite their respective sines, tangents, &c. The uses of such an alteration would be many and great, but are too obvious and important to need pointing out or enforcing. I have therefore had for a long time a great desire to commence this arduous task; but continual interruptions have hitherto prevented me from making any considerable progress, in so desirable an undertaking. But I am not without hopes that some future occasion may prove more propitious to my ardent wishes. It is not however to be expected, that this work can be accomplished by the labours of one person only; it will require rather the united endeavours of many. I shall therefore explain a few particulars relative to my project of this work, with a view to obtain from others, who may have leisure and abilities for it, their kind assistance, either by communicating hints of improvements, or by undertaking some part of the computations, to which they may be excited by their zeal for the accomplishment of so important a work, and by the extreme facility with which the calculations in this way are made.

2. In the first place then I would observe, that I think it will be sufficient to print the sines, tangents, &c, to 7 places of figures; and that therefore it will be necessary to compute them to 10 places, in order effectually to secure the truth of the 7th place to the nearest unit.

3. I would assume the radius equal to 100000, or suppose it to be divided into 100000 equal parts. Then it is well known, that the semi-circumference will be 314159·26536 nearly, and consequently the quadrant nearly 157079·63268 of the same equal parts, which is less than 157080 by ·36732, or nearly $\frac{1}{3}$ of a unit, or nearer $\frac{3}{8}$ = ·375, or nearer $\frac{4}{11}$ = ·3636, or still nearer $\frac{7}{19}$ = ·3684, or still nearer $\frac{11}{30}$ = ·36666 &c. And the half quadrant, or $\frac{1}{8}$ of the circle, 78539·81634 which is less than 78540 by only ·18366, or nearly $\frac{1}{2}$ only of any of the above-mentioned fractions.

4. The table may consist of 5 or more columns; the first column to contain the regular arithmetical series of arcs, differing by unity, from the beginning, in this manner, 1, 2, 3, 4, 5, &c, up to half the quadrant, the next less whole number being 78539; then for the higher numbers, or those in the latter half quadrant, besides adding 1 continually, there must be at the first added the decimal ·63268, which will make all the numbers in this half become the exact complements of the first half, which consists of whole numbers only; and these will be the lengths of the arcs. Or, in order to include the quadrantal arc 78539·81634, the first column may be continued up to 78540. The second column to contain the corresponding degrees, minutes and seconds, to the nearest second, or to the true seconds and decimals of a second, for the convenience of easily changing the tables from the one measure to the other, or to make them answer to both methods; and the 3d, 4th, 5th, &c columns, to contain the corresponding sines, tangents, secants, &c.

5. The tables may be disposed as at present, namely, continuing them downwards by the left-hand side of the pages, as far as to the middle of the quadrant, and then returning them again backwards and upwards by the right-hand side of the pages.

6. In this disposition, the numbers on the same line, the one on the left and the other on the right, will be exact complements of each other to a quadrant, and the decimal

·63268, in every number in the latter half quadrant, in each page, may be placed either at the bottom of the column, or lengthways on the sides of it.

7. A specimen of the first page of the table will therefore be as follows:

Arcs	° ′ ″	Sines.	Tang.	Secants.	Cosec.	Cotan.	Cosin.	° ′ ″	Arcs.
0	0 0 0	00000·00	00000·00	100000·00	infin.	infin.	100000·00	90 0 0	79
1	2	1·00							78
2	4	2·00							77
3	6	3 00							76
4	8	4 00							75
&c									&c
·									
·									
·									
·									·
80									
·									
·									
·									
·									
&c									&c
96									83
97									82
98									81
99									80
Arcs	° ′ ″	Cosin.	Cotan.	Cosec.	Secants.	Tang.	Sines.	° ′ ″	Arcs.

+ 632679 &c

8. To fill up the second column: Since the length of the quadrant is 157079·63267948966, and the number of seconds in the quadrant is $90 \times 60 \times 60 = 324000$; therefore, as 157079·632 &c.: 324000 :: 1 : 2·062648062470964 = the number of seconds answering to each unit in our division of the quadrant, and which therefore, being continually added, will fill up the second column.

9. The number of seconds to be always added, being 2, and the decimal ·062648062470964, which is nearly equal to $\frac{1}{16}$, for $\frac{1}{16}$ is ·0625; therefore, besides adding 2 every time, we must also add 1″ more at every 16, which will make 3″ to be added at every 16th time, and 2″ at every other time besides; but the first time the 3″ must be added will be at the arc or number 8, to have them to the nearest second, the repetition of the fraction at the arc 8 amounting

to above $\frac{1}{2}$ a second; and then the 3″ must be added at every 16 afterwards, viz, at 24, 40, 56, 72, 88, 104, &c.

10. But, besides the constant addition of 2″ every time, and of 1″ more every 16th time, there must be 1″ more added for every 6753½ time, on account of the excess of the fraction ·062648062470964 over the fraction ·0625 or $\frac{1}{16}$: for that excess is ·000148062470964, which is $= \frac{1}{6753\frac{1}{2}}$. And the easiest method of making this last addition of 1 at every 6753½, will be to make the increase of the 1, on account of the $\frac{1}{16}$, at a unit sooner for every $422\frac{3}{32}$; because 16 is $422\frac{3}{32}$ times contained in 6753½; by which means the incremental units, for the $\frac{1}{16}$, will become 1 more at that number 6753½, which last unit may be considered as the increment of the former increment for the $\frac{1}{16}$; and so proceed up to the quadrant; which will complete the second column of arcs to the nearest second in each number. Or this second column may be exactly computed to as many decimals as we please, by adding continually the 2″ and decimals, viz, 2·062648062470964. But at the middle of the quadrant, where the numbers return again upwards by the right-hand, there will for once be to be added only the seconds and decimals answering to the arc ·63268, viz, 1·30499618 seconds, that number being necessary to make the numbers on the right-hand to be the exact complements of those on the left. Or it will perhaps be proper to make them to the nearest unit, in the 6th place of decimals. And to fill up the second column to this degree of accuracy, add continually 2·062648 seconds, but at the 9th line add 1 more, or 2·062649, because 9 times 062470964 amounts to 56223867, or more than half a unit at that place; and after that add 1 more than 2·062648 at every 16th line, viz, at 25, 41, 57, 73, &c, because 16 times ·062470964 amounts to ·99953542, or nearly 1, it being only ·00046458 less than 1. And this number ·00046458, thus added too much, will, in 134 times adding it, amount to more than 06223867, the excess of 56223867 above 5, or half a unit,

at that place; therefore at the line or number 2153, or $9 + 16 \times 134$, which would be to have the 1 more added, let the 1 be there omitted, and add it at the next line or 2154, the true decimals after the first six, for 2153, being 499985, and for 2154 they are 562456. Continue thus always adding 1 more at every 16th line, except at the following numbers, where the 1 must be omitted, and added at the next following number; viz,

2153	12926	23683	34456	45213	55970		
4314	15071	25844	36601	47358	58131	66743	73194
6459	17232	27989	38746	49519	60276	68888	75339
8620	19377	30150	40907	51664	62437	71033	77500
10765	21538	32295	43052	53825	64582		

And thus proceed to the middle of the quadrant; by which means all the numbers will be to the nearest unit in the sixth or last place. Also, to have a check upon these numbers at certain intervals, it may be proper to proceed in this manner: First find every 100th number, by adding its decimal ·264806 &c, verifying them at every 10th; then find every 16th number, by adding continually ·002369 &c, which will also be checked and verified at every 25th addition by one of the former set of 100, for 25 times 16 make 400, using a proper precaution to preserve each number true to the nearest unit in the 6th or last decimal.

As to the decimals of the numbers in the latter half of the quadrant, they will be the complements, to 1, of the corresponding numbers in the first half; and therefore they may be all easily found by taking each figure from 9, and the last from 10. But it will be safest to find only every 10th decimal in this way, and to fill up the intermediate nine by adding, as before, the constant decimal 062648; by which means they will be checked and verified at every 10th number.

11. To fill up the third column, or that of sines, as well as those of tangents and secants, it may first be observed, that the old tables of those lines to every minute, or even to

every ten seconds of the quadrant, cannot be of so much use as it might seem at first sight; as the very near coincidence of the numbers in the new and old divisions appear very seldom to happen. I find indeed, that our arc 1309 answers nearly to 45 minutes, that arc exceeding 45′ by only ·00632363 or $\frac{1}{158}$ part of a second nearly, and so in proportion for their equimultiples. But though this degree of coincidence may be sufficient for checking the corresponding values of the arcs, in the first and second columns, we are not thereby authorised to consider the sine, tangent, or secant, of 1309, as accurately equal to that of 45′, in all the seven places of figures, but differing from it by nearly the $\frac{1}{158}$ part of the difference corresponding to 1″, which is about $\frac{1}{3}$ of a unit in the sines and tangents, though next to nothing in the secants. This therefore, though it makes no sensible difference in this particular case, will cause a difference that must not be neglected, in the equi-multiples of 1309 and 45′, the sines and tangents of which will differ by half a unit or more, and therefore will not be expressed by the same number, but will have some small difference in the seventh or last figure. And the same will happen in almost all the other arcs; so that generally the sines, &c, which are exact for the arcs in the first column, will not be quite so for those in the second, when expressed in whole seconds only, since these will sometimes differ by the part corresponding to almost half a second. However, in this, or any other case, where the difference is exactly known, we may profitably make use of the numbers in the old tables, for constructing or verifying those of the new, by taking in the proportional part of the difference. Let, therefore, all the sines, &c, of every 1309 be computed from the old tables, and entered in the new, by adding to the sine, &c, of the corresponding multiple of 45′, the like multiple of the $\frac{1}{158}$ part of the proportional difference for 1″. This will give about 120 sines, &c, to serve as a verification of the computations by the more general methods. But if the second column be exactly constructed,

with all its decimal places, by the continual addition of 2·06264807, the old tables may be converted into the new, by allowing for the odd seconds and decimals. And for this purpose it will, perhaps, be best to use the large table of Rheticus, which contains the sines, tangents, and seconds, to ten places of figures for every 10″, and also the differences. At least, such sines, &c, may be found in this way, as have their seconds and decimals well adapted for the purpose; and for such as would be found too troublesome in this way, recourse may be had to some of the following methods.

12. Let us now examine the expressions for the sines, &c, by infinite series.

The radius being 1, and arc a, it is well known that the

$$\begin{array}{lll}
\text{sine is} & = a & -\tfrac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7 + \frac{1}{362880}a^9 - \text{\&c.} \\
\text{cosine} & = 1 & -\tfrac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6 + \frac{1}{40320}a^8 - \text{\&c.} \\
\text{tang.} & = a & +\tfrac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 + \frac{62}{2835}a^9 + \text{\&c.} \\
\text{cotang.} & = a^{-1} & -\tfrac{1}{3}a - \frac{1}{45}a^3 - \frac{2}{945}a^5 - \frac{1}{4725}a^7 - \text{\&c.} \\
\text{secant} & = 1 & +\tfrac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6 + \frac{277}{8064}a^8 + \text{\&c.} \\
\text{cosec.} & = a^{-1} & +\tfrac{1}{6}a + \frac{7}{360}a^3 + \frac{31}{15120}a^5 + \frac{127}{604800}a^7 + \text{\&c.}
\end{array}$$

Or the same series are thus otherwise expressed:

$$\begin{array}{ll}
\text{sine} & = a - \frac{1}{2 \,.\, 3}a^3 + \frac{b}{4 \,.\, 5}a^5 - \frac{c}{6 \,.\, 7}a^7 + \frac{d}{8 \,.\, 9}a^9 - \text{\&c.} \\
\text{cosine} & = 1 - \frac{1}{2}a^2 + \frac{b}{3 \,.\, 4}a^4 - \frac{c}{5 \,.\, 6}a^6 + \frac{d}{7 \,.\, 8}a^8 - \text{\&c.} \\
\text{tangent} & = a + \frac{1}{1 \,.\, 3}a^3 + \frac{8b}{4 \,.\, 5}a^5 + \frac{17c}{6 \,.\, 7}a^7 + \frac{29\frac{3}{17}d}{8 \,.\, 9}a^9 + \text{\&c.} \\
\text{cotang.} & = a^{-1} - \frac{1}{3}a - \frac{b}{15}a^3 - \frac{2c}{21}a^5 - \frac{d}{10}a^7 - \text{\&c.} \\
\text{secant} & = 1 + \frac{1}{2}a^2 + \frac{5b}{12}a^4 + \frac{61c}{150}a^6 + \frac{1385d}{3416}a^8 + \text{\&c.} \\
\text{cosec.} & = a^{-1} + \frac{1}{6}a + \frac{7b}{60}a^3 + \frac{31c}{294}a^5 + \frac{127d}{1240}a^7 + \text{\&c.}
\end{array}$$

where b, c, d, &c, denote the preceding co-efficients. And hence, with the help of the table of the first ten powers of the first 100 numbers, in p. 101 of my tables of powers, published by order of the Board of Longitude, may be easily found the sines, &c, of all arcs up to 100, by only dividing those powers by their respective co-efficients, as also of all multiples of these arcs by 10, 100, &c, by only varying the decimal points in the several terms, as the figures will be all the same: and thus a number of primary sines, &c, may be found, to check or verify the same, when computed by other methods. By this means will be found the sines, &c, of the arcs

1, 10, 100, 1000, 10000, 100000;
2, 20, 200, 2000, 20000;
3, 30, 300, 3000, 30000;
4, 40, 400, 4000, 40000;
&c, till
99, 990, 9900, 99000, 990000.

13. Again, it is evident, that, of the terms in the series for the sine, the first term a alone will give the sine true to the nearest unit in the 9th place, in the first 144 sines, or the arc and sine will be the same for nine places, as far as the arc 144; but they will agree to the nearest unit in the 7th place, as far as the arc 669; after which, the second term of the series must be included.

14. When the second term is taken in, these two terms, $a - \frac{1}{6}a^3$, will give the sines true to the nearest unit in the 9th place, till the arc becomes 3500. Now the numbers in my table of cubes, published by order of the Board of Longitude, extend to 10000, and therefore all the above cubes are found in it; consequently, taking the 6th part of those cubes, and subtracting it from the corresponding arcs, the remainders will be the sines of those arcs, as far as till the arc be 3500: after which the third term of the series may be taken in, or other methods may be used.

15. But since, for any arc a, this is a general theorem,

viz, as radius : 2 cos. a :: sin. na : sin. $(n-1) \times a$ + sin. $(n+1) \times a$; taking $a = 1$, radius 10000, the sine of a will be $1 - \cdot 0000000000\frac{1}{6}$, and the cosine of a will be $100000 - \cdot 000005$; then the above proportion will become $100000 : 200000 - \cdot 00001$, or $1 : 2 - \cdot 0000000001$ :: sin. n : sin. $(n-1)$ + sin. $(n+1)$; consequently sin. $(n-1)$ + sin. $(n+1)$ is $=$ 2 sin. $n - \cdot 0000000001$ sin. n, and the sines are in arithmetical progression except only for the small difference of $\cdot 0000000001$ sin. n; hence sin. $(n+1)$ is $= (2 - \cdot 0000000001) \times$ sin. n − sin. $(n-1)$; and therefore, taking n successively equal to 1, 2, 3, 4, &c, the series of sines will be as follows:

$$\begin{aligned}
\sin. 1 &= 1 - \cdot 0000000000\tfrac{1}{6}; \\
\sin. 2 &= (2 - \cdot 0000000001) \times \sin. 1; \\
\sin. 3 &= (2 - \cdot 0000000001) \times \sin. 2 - \sin. 1; \\
\sin. 4 &= (2 - \cdot 0000000001) \times \sin. 3 - \sin. 2; \\
\sin. 5 &= (2 - \cdot 0000000001) \times \sin. 4 - \sin. 3; \\
&\text{\&c.}
\end{aligned}$$

And by this theorem, viz, sin. $(n+1) = (2 - \cdot 0000000001) \times$ sin. n − sin. $(n-1)$, may easily be filled up the intervals between those primary numbers mentioned in former articles.

16. In like manner, as radius : 2 cos. a :: cos. na : cos. $(n-1) \cdot a$ + cos. $(n+1) \cdot a$; and hence this theorem, cos. $(n+1) = (2 - \cdot 0000000001) \times$ cos. n − cos. $(n-1)$, by which the cosines will be all easily filled up. And these two theorems, for the sines and cosines, are so easy and accurate, that we need not have recourse to any other, but only to check and verify these at certain intervals, as at every 100th number, by a proportion from Rheticus's canon, as mentioned at Art. 11, or by any other way.

17. The sines and cosines being completed, the difference between the radius and cosine will be the versed sine; the difference between radius and sine will be the co-versed sine; and the sum of the radius and cosine will be the supplement versed sine.

18. From the sines and cosines also, the tangents, cotangents, secants, and cosecants, may be made by these known proportions, viz, as

1. cosine	: radius	:: sine	: tangent,
2. sine	: radius	:: cosine	: cotangent,
3. cosine	: radius	:: radius	: secant,
4. sine	: radius	:: radius	: cosecant,
5. radius	: sine	:: secant	: tangent,
6. radius	: cosine	:: cosecant	: cotangent,
7. tangent	: radius	:: radius	: cotangent.

Therefore, the reciprocal of the cosine will be the secant; the reciprocal of the sine, the cosecant; the quotient of the sine by the cosine, the tangent; and the quotient of the cosine by the sine, the cotangent; or the product of the sine and secant will be the tangent, and the product of the cosine and cosecant, the cotangent; or, lastly, the reciprocal of the tangent is the cotangent; proper regard being had to the number of decimals, on account of our radius being 100000, instead of 1 only.

And these are to be used when the application happens to be easier than the general series, and easier than by proportion from Rheticus's canon.

But there are other particular theorems, which, by a little address, may be rendered more expeditious than any of the former: thus,

19. In any two arcs, this is a general proportion;
As the difference of their sines:
to the sum of their sines ::
so tangent of half the difference of the arcs :
to tangent of half their sum.

So that, by taking continually the arcs, having the common difference 2, the third term of this proportion will be 1, and the fourth term will be found by dividing the sum of the sines by their difference, which divisor, or difference, will

never consist of more than four or five figures, viz, about half the number of figures that are in the divisors mentioned in the preceding article.

20. Again, As the difference of the cosines :
to the sum of the cosines : :
so tangent of half their difference :
to cotangent of half their sum.

And thus the cotangents will be found, by dividing the sum of the cosines of two arcs, differing by 2, by their small difference.

21. Also, the secant of an arc, is equal to the sum of its tangent and the tangent of half its complement; and the cosecant of an arc, is equal to the sum of its cotangent and the tangent of half the arc; or half the sum of the tangent and cotangent, is equal to the cosecant of the double arc. Whence the secants and cosecants will be easily made. And thus we have pointed out methods, by which the whole tables may be readily constructed.

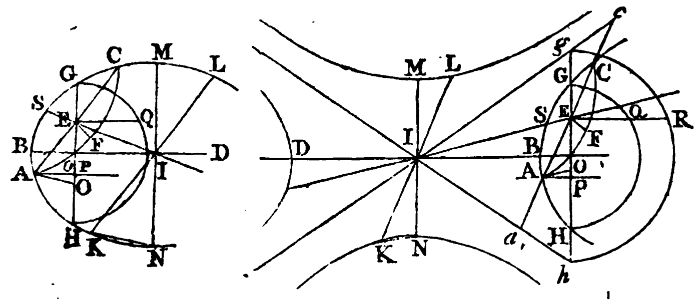
TRACT XXX.

ON THE SECTIONS OF SPHEROIDS AND CONOIDS.

PROPOSITION I.

If any solid, formed by the rotation of a conic section about its axe, i. e. *a spheroid, paraboloid, or hyperboloid, be cut by a plane in any position; the section will be some conic section, and all the parallel sections will be like and similar figures.*

Demon. Let ABC be the generating section, or a section of the given solid through its axe BD, and perpendicular to the proposed section AFC, their common intersection being AC; let GH be any other line meeting the generating section in G and H, and cutting AC in E; and erect EF perpendicular to the plane ABC, and meeting the proposed plane in F.



Now, if AC and GH be conceived to be moved continually parallel to themselves, then will the rectangle $AE \times EC$ be to the rectangle $GE \times EH$, always in a constant ratio; but if GH be perpendicular to BD, the points G, F, H will be in the circumference of a circle whose diameter is GH, so that $GE \times EH$ will be $= EF^2$; therefore $AE \times EC$ will be to EF^2,

always in a constant ratio; consequently AFC is a conic section, and every section parallel to AFC will be of the same kind with it, and similar to it. Q. E. D.

Corol. 1. The above constant ratio, in which $AE \times EC$ is to EF^2, is that of KI^2 to IM^2, the squares of the diameters of the generating section respectively parallel to AC, GH; that is, the ratio of the square of the diameter parallel to the section, to the square of the revolving axe of the generating plane.—This will appear by conceiving AC and GH to be moved into the positions KL, MN, intersecting in I, the centre of the generating section.

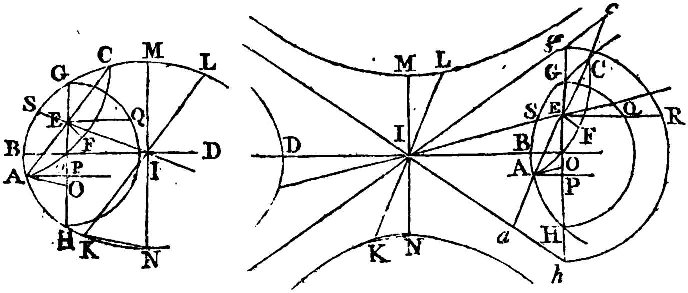
Corol. 2. And hence it appears, that the axes AC and 2EF of the section, supposing E now to be the middle of AC, will be to each other, as the diameter KL is to the diameter MN of the generating section.

Corol. 3. If the section of the solid be made so as to return into itself, it will evidently be an ellipse. Which always happens in the spheroid, except when it is perpendicular to the axe; which position is also to be excepted in the other solids, the section being always then a circle: in the paraboloid the section is always an ellipse, excepting when it is parallel to the axe: and in the hyperboloid the section is always an ellipse, when its axe makes with the axe of the solid, an angle greater than that made by the said axe of the solid and the asymptote of the generating hyperbola; the section being an hyperbola in all other cases, but when those angles are equal, and then it is a parabola.

Corol. 4. But if the section be parallel to the fixed axe BD, it will be of the same kind with, and similar to, the generating plane ABC; that is, the section parallel to the axe, in a spheroid, is an ellipse similar to the generating ellipse; in the paraboloid, the section is a parabola similar to the generating parabola; and in an hyperboloid, it is an hyperbola similar to the generating hyperbola of the solid.

Corol. 5. In the spheroid, the section through the axe is

the greatest of the parallel sections; but in the hyperboloid, it is the smallest; and in the paraboloid, all the sections parallel to the axe are equal to one another.—For, the axe is the greatest parallel chord line in the ellipse, but the least in the opposite hyperbolas, and all the diameters are equal in a parabola.



Corol. 6. If the extremities of the diameters KL, MN, be joined by the line KN, and AO be drawn parallel to KN, and meeting GEH in O, E being the middle of AC, or AE the semi-axe, and GH parallel to MN. Then EO will be equal to EF, the other semi-axe of the section.—For, by similar triangles, KI : IN :: AE : EO.

Or, upon GH as a diameter, describe a circle meeting EQ, perpendicular to GH, in Q; and it is evident that EQ will be equal to the semi-diameter EF.

Corol. 7. Draw AP parallel to the axe BD of the solid, and meeting the perpendicular GH in P. Then it will be evident that, in the spheroid, the semi-axe EF = EO will be greater than EP; but in the hyperboloid, the semi-axe EF = EO, of the elliptic section, will be less than EP; and in the paraboloid, EF = EO is always equal to EP.

SCHOLIUM.

The analogy of the sections of an hyperboloid to those of the cone, are very remarkable, all the three conic sections being formed by cutting an hyperboloid in the same position as the cone is cut.

Thus, let an hyperbola and its asymptote be revolved together about the transverse axe, the former describing an hyperboloid, and the latter a cone circumscribing it; now let them be supposed to be both cut by one plane in any position, then the two sections will be like, similar, and concentric figures: that is, if the plane cut both sides of each, the sections will be concentric, similar ellipses; if the cutting plane be parallel to the asymptote, or to the side of the cone, the sections will be parabolas; and in all other positions, the sections will be similar and concentric hyperbolas.

That the sections are like figures, appears from the foregoing corollaries. That they are concentric, will be evident when we consider that Cc is $= Aa$, producing AC both ways to meet the asymptotes in a and c. And that they are similar, or have their transverse and conjugate axes proportional to each other, will appear thus: Produce GH both ways to meet the asymptotes in g and h; and on the diameters GH, gh, describe the semi-circles GQH, gRh, meeting EQR, drawn perpendicular to GH, in Q and R; EQ and ER being then evidently the semi-conjugate axes, and EC, Ec, the semi-transverse axes of the sections. Now if GH and AC be conceived to be moved parallel to themselves, $AE \times EC$ or CE^2, will be to $GE \times EH$ or EQ^2, in a constant ratio, or CE to EQ will be a constant ratio; and since cE is as Eg, and aE as Eh, $aE \times Ec$ or cE^2, will be to $gE \times Eh$ or ER^2, in a constant ratio, or cE to ER will be a constant ratio; but at an infinite distance from the vertex, C and c coincide, or $EC = Ec$, as also $EG = Eg$, consequently EQ is then $=$ ER, and CE to EQ will be $= c$E to ER; but as these ratios are constant, if they be equal to each other in one place, they must be always so; and consequently $CE : Ec :: QE : ER$.

And this analogy of the sections will not seem strange, when we consider that a cone is a species of the hyperboloid; or a triangle a species of the hyperbola, whose axes are infinitely small.

PROPOSITION II.

If SI *be the semi-diameter belonging to the double ordinate* AEC *of the generating plane,* AEC *being the diameter of the section* AFC, *conceived to be moved continually parallel to itself; and if x denote any part of the diameter* SI, *intercepted by* E *the middle of* AC, *and any given fixed point taken in* SI; *then will the section* AFC *be always as $a + bx + cxx$; a, b, c, being constant quantities; b in some cases affirmative, and in others negative; c being affirmative in the hyperbola, and negative in the ellipse, and nothing in the parabola; and a may always be supposed to denote the distance of the given fixed point from the vertex* S.

Demon. In any conic section, AC^2 is as $a + bx + cxx$; but all the parallel sections are like and similar figures, and similar plane figures are as the squares of their like dimensions; therefore the section AFC is as AC^2, that is, as $a + bx + cxx$. Q. E. D.

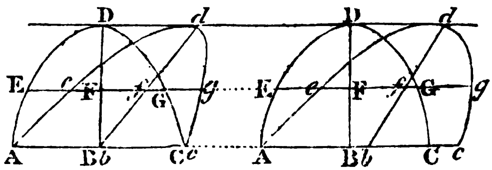
Corollary. If the given fixed point, where x begins, coincide with the vertex S, then will a be equal to nothing, and the section will be as $bx \pm cxx$, or as $x \pm dxx$, in the hyperbola and ellipse, and as bx, or as x, in the parabola.

TRACT XXXI.

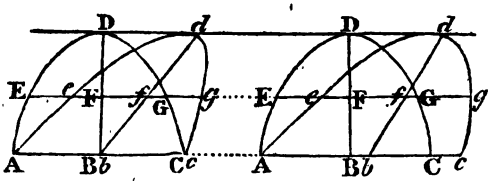
COMPARISON OF CURVES.

PROPOSITION I.

If ADC, A*dc* *be two curves of any one and the same species, and there be drawn* D*d* *a tangent to them both; and from the points of contact* D, *d*, *there be drawn the diameters* DFB, *dfb*, *bisecting the double ordinates* EG *and* AC, *eg and* A*c*, *which are parallel to* D*d*: *then, if the corresponding ordinates* EG *and* eg, *or* AC *and* A*c*, *be equal to each other, or in any ratio to each other in any one part, they will always be equal or in the same constant ratio to each other in every part.*



Demon. For, let x denote any abscissa DF. Then, since df is always to DF in a constant ratio, let that ratio be that of r to 1; then will df be denoted by rx. Also let y denote the double ordinate EG, and z the ordinate eg; and let the general equation $y = ax + bx^2 + cx^3 + dx^4$ &c, express the general relation of that species of curve, or the relation of the ordinate to the abscissa; then will the equation $z = \text{A}rx + \text{B}r^2x^2 + \text{C}r^3x^3 + \text{D}r^4x^4$ &c, express the general relation between df and eg. Now, in order to determine the relation of the fixed constant coefficients of the terms of these two series, let the corresponding ordinates be conceived to flow into that situation where they are equal or have a known ratio, as at AC, A*c*, and let their ratio there



be that of n to 1, viz, $ac : AC :: n : 1$, or $ac = n \times AC$, or $z = ny$ at that place; therefore then also is $nax + nbx^2 + ncx^3$ &c $= Arx + Br^2x^2 + Cr^3x^3$ &c; hence, by equating the like terms of this equation, we have $na = Ar$, $nb = Br^2$, $nc = Cr^3$, &c; and since these are all *constant* quantities, these equations will express their *constant* or *general* relation: let now na, nb, nc, &c, be substituted instead of their values Ar, Br^2, Cr^3, &c, in the general series expressing the value of z, and it will be always

$$z = nax + nbx^2 + ncx^3 \text{ \&c};$$

but it is always - $y = ax + bx^2 + cx^3$ &c;

therefore $z = ny$, or $z : y :: n : 1$, i. e. $eg : EG :: ac : AC$.

Again, when ac is once $= AC$, or $n = 1$, then always $z = y$, or it is always $eg = EG$. Q. E. D.

PROPOSITION II.

The two curves of the same kind being still related as specified in the last proposition: then, the corresponding areas AEGC, *Aegc, or* ADC, *Adc, or* EDG, *edg, will always be equal to each other, or in the same constant ratio with the ordinates.*

Demon. If the corresponding ordinates be supposed to flow from any position, always parallel to themselves: then, since the fluxion of the area is equal to the ordinate drawn into the fluxion of the abscissa; and, the fluxions of the abscissas are equal, or in a constant ratio, the fluxions of the areas will be in a constant ratio also; but when two fluxions are in a constant ratio, their fluents are likewise in the same constant ratio; therefore the areas generated are as the generating ordinates; but the ordinates are either equal or in a constant ratio; therefore the areas are also equal or in the same constant ratio. Q. E. D.

TRACT XXXII.

THEOREM FOR THE CUBE ROOT OF A BINOMIAL.

IF - - $\sqrt[3]{(a+\sqrt{b})}$ be $=p+\sqrt{q}$;
then shall $\sqrt[3]{(a-\sqrt{b})}$ be $=p-\sqrt{q}$. And *vice versa*.

Demon. For, since $\sqrt[3]{(a+\sqrt{b})}=p+\sqrt{q}$,
therefore - $a+\sqrt{b}=p^3+3p^2\sqrt{q}+3pq+q\sqrt{q}$;
hence - $a=p^3+3pq$,
and - - $\sqrt{b}=3p^2\sqrt{q}+q\sqrt{q}$.
Consequently, by subtraction, it is

$$a-\sqrt{b}=p^3-3p^2\sqrt{q}+3pq-q\sqrt{q},$$

and the cubic root of this is $\sqrt[3]{(a-\sqrt{b})}=p-\sqrt{q}$. Q. E. D.

Again, if - - $\sqrt[3]{(a-\sqrt{b})}$ be $=p-q$;
then shall - - $\sqrt[3]{(a+\sqrt{b})}$ be $=p+q$.
This is proved like the former, by adding, instead of subtracting.

Corol. Hence $\sqrt[3]{(a+\sqrt{b})}+\sqrt[3]{(a-\sqrt{b})}=2p$ the root of a cubic equation.

The necessity or occasion for a rule to extract the cube root of such binomials as $a+\sqrt{b}$ and $a-\sqrt{b}$, occurred as early as the first discovery of the rules for solving cubic equations by Tartalea, since one of the rules is expressed in this form, $\sqrt[3]{(a+\sqrt{b})}+\sqrt[3]{(a-\sqrt{b})}$. On this occasion Tartalea invented the following rule, as stated at large in the next following Tract, on the History of Algebra. The rule is this, and will hold good in all such cases as will admit of a perfect cubic root. "Take," says Tartalea, "either of the two terms of the binomial, and divide it into two such parts, that one of them may be a complete cube, and the other part exactly divisible by 3; then the cube root of the said cubic part will be one term of the required root; and the square root of the quotient arising from the division of $\frac{1}{3}$ of the second part by the said cube

root of the former, will be the other member of the root sought." But this rule will perhaps be better understood in characters thus: Let m be one member of the given binomial, whose cube root is sought; and let it be divided into the two parts a^3 and $3b$, so that $a^3 + 3b$ be $= m$; then is $a + \sqrt{\frac{b}{a}}$ the cube root of the proposed quantity, if it has a root. Thus, in the quantity $\sqrt{108} + 10$, taking the term $\sqrt{108}$ for m, this divides into the two equal parts $\sqrt{27}$ and $\sqrt{27}$, making $a^3 = \sqrt{27}$, and $3b = \sqrt{27}$ also; hence $a = \sqrt{3}$, and $b = \sqrt{3}$ also; consequently $a + \sqrt{\frac{b}{a}} = \sqrt{3} + \sqrt[4]{\frac{3}{3}}$ or $\sqrt{3} + 1$, for the cube root of the binomial sought. Again, taking the term 10; this divides into 1 and 9, where $a^3 = 1$ or $a = 1$, and $3b = 9$, or $b = 3$; therefore $a + \sqrt{\frac{b}{a}}$ becomes $1 + \sqrt{3}$ for the cube root of $\sqrt{108} + 10$, the same as before.

"And thus," Tartalea adds, "we may know whether any proposed binomial or residual be a cube or a noncube; for if it be a cube, the same two terms for the root must arise from both the given terms separately; and if the two terms of the root cannot thus be brought to agree both ways, such binomial or residual will not be a cube."

If therefore the expression for the root of a cubic equation should lead to the expression $\sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}$; then since, as above $\sqrt[3]{(\sqrt{108} + 10)}$ is $= \sqrt{3} + 1$; and that, by our theorem, $\sqrt[3]{(\sqrt{108} - 10)}$ is $= \sqrt{3} - 1$; therefore $(\sqrt{3} + 1) - (\sqrt{3} - 1) = 2$ is the root of the equation sought.

TRACT XXXIII.

HISTORY OF ALGEBRA.

Algebra is usually understood to be a general method of resolving mathematical problems by means of equations. Or, it is a method of performing the calculations of all sorts of quantities by means of general signs or characters. At first, numbers and things were expressed by their names at full length; but afterwards these were abridged, and the initials of the words used instead of them; and, as the art advanced further, the letters of the alphabet came to be employed as general representations of all sorts of quantities. Other marks were also gradually introduced, to express all sorts of operations and combinations; so as to entitle it to different appellations—as, universal arithmetic, and literal arithmetic, and the arithmetic of signs.

The etymology of the name, Algebra, is given in various ways. It is, however, pretty generally considered, that the word is Arabian, and that from those people we had the name, as well as the art itself, as is testified by Lucas de Burgo, the first European author whose treatise was printed on this art, and who also refers to former authors and masters, from whose writings he had learned it. The Arabic name he gives it, is Alghebra e Almucabala, which is explained to signify the art of restitution and comparison, or opposition and restoration, or resolution and equation, all which agree well enough with the nature of this art. Some however derive it from various other Arabic words; as from Geber, a celebrated philosopher, chemist, and mathematician, to whom also they ascribe the invention of this science; and some derive it from the word Geber, which with the particle *al*, makes Algeber, which is purely Arabic,

and signifies the reduction of broken numbers or fractions to integers.

But Peter Ramus, in the beginning of his algebra, says "the name algebra is Syriac, signifying the art and doctrine of an excellent man. For Geber, in Syriac, is a name applied to men, and is sometimes a term of honour, as master or doctor among us. That there was a certain learned mathematician, who sent his algebra, written in the Syriac language, to Alexander the Great, and he named it Almucabala, that is, the book of dark or mysterious things, which others would rather call the doctrine of algebra. And to this day the same book is in great estimation among the learned in the oriental nations, and by the Indians who cultivate this art it is called aljabra, and alboret; though the name of the author himself is not known." But Ramus gives no authority for this singular paragraph. It has however on various occasions been distinguished by other names. Lucas Paciolus, or de Burgo, in Italy, called it l'Arte Magiore: ditta dal vulgo la Regola de la Cosa over Alghebra e Almucabala; calling it l'Arte Magiore, or the greater art, to distinguish it from common arithmetic, which is called l'Arte Minore, or the lesser art. It seems too that it had been long and commonly known in his country by the name Regola de la Cosa, or Rule of the Thing; whence came our rule of coss, cosic numbers, and such like terms. Some of his countrymen followed his denomination of the art; but other Italian and Latin writers called it Regula rei & census, the rule of the thing and the product, or the root and the square, as the unknown quantity in their equations commonly ascended no higher than the square or second power. From this Italian word *census*, pronounced *chensus*, came the barbarous word *zenzus*, used by the Germans and others, for quadratics; with the several zenzic or square roots. And hence ℞, ʒ, ȼ, which are derived from the letters *r*, *z*, *c*, the initials of *res*, *zenzus*, *cubus*, or root, square, cube, came to be the signs or characters of these

words: like as ℞ and $\surd$, derived from the letters R, r, became the signs of radicality.

Later authors, and other nations, used some the one of those names, and some another. It was also called specious arithmetic by Vieta, on account of the species, or letters of the alphabet, which he brought into general use; and by Newton it was called universal arithmetic, from the manner in which it performs all arithmetical operations by general symbols, or indeterminate quantities.

Some authors define algebra to be the art of resolving mathematical problems: but this is the idea of analysis, or the analytic art in general, rather than of algebra, which is only one particular species of it.

Indeed algebra properly consists of two parts: first, the method of calculating magnitudes or quantities, as represented by letters or other characters; and secondly the manner of applying these calculations in the solution of problems.

In algebra, as applied to the resolution of problems, the first business is to translate the problem out of the common into the algebraic language, by expressing all the conditions and quantities, both known and unknown, by their proper characters, arranged in an equation, or several equations if necessary, and treating the unknown quantity, whether it be number, or line, or any other thing, in the same way as if it were a known one: this forms the composition. Then the resolution, or analytic part, is the disentangling the unknown quantity from the several others with which it is connected, so as to retain it alone on one side of the equation, while all the other, or known quantities, are collected on the other side, and so giving the value of the unknown one. And as this disentangling of the quantity sought, is performed by the converse of the operations by which it is connected with the others, taking them always backwards in the contrary order, it hence becomes a species of the analytic art, and is called the modern analysis, in

contradistinction to the ancient analysis, which chiefly respected geometry, and its applications.

There have arisen great controversies and sharp disputes among authors, concerning the history of the progress and improvements of algebra; arising partly from the partiality and prejudices which are natural to all nations, and partly from the want of a closer examination of the works of the older authors on this subject. From these causes it has happened, that the improvements made by the writers of one nation, have been ascribed to those of another; and the discoveries of an earlier author, to some one of much later date. Add to this also, that the peculiar methods of many authors have been described so little in detail, that our information derived from such histories, is but very imperfect, and amounting only to some general and vague ideas of the true state of the arts. To remedy this inconvenience therefore, and to reform this article, I have taken the pains carefully to read over in succession all the older authors on this subject, which I have been able to meet with, and to write down distinctly a particular account and description of their several compositions, as to their contents, notation, improvements, and peculiarities; from the comparison of all which, I have acquired an idea more precise and accurate than it was possible to obtain from other histories, and in a great many instances very different from them. The full detail of these descriptions would employ a volume of itself, and would be far too extensive for this place: I must therefore limit this article to a very brief abridgment of my notes, remarking only the most material circumstances in each author; from which a general idea of the chain of improvements may be perceived, from the first rude beginnings, down to the more perfect state; from which it will appear that the discoveries and improvements made by any one single author, are scarcely ever either very great or numerous; but that, on the contrary, the improvements are almost always very slow and gradual, from former writers,

successively made, not by great leaps, and after long intervals of time, but by gradations which, viewed in succession, become almost imperceptible.

OF DIOPHANTUS'S ALGEBRA.

As to the origin of the analytic art, of which algebra is a species, it is doubtless as old as any science in the world, being the natural method by which the mind investigates truths, causes, and theories, from their observed effects and properties. Accordingly, traces of it are observable in the works of the earliest philosophers and mathematicians, the subject of whose enquiries most of any require the aid of such an art. And this process constituted their analytics. Of that part of analytics, however, which is properly called algebra, the oldest treatise which has come down to us, is that of Diophantus of Alexandria, who flourished about the year 150 after Christ, and who wrote, in the Greek language, thirteen books of algebra or arithmetic, as mentioned by himself at the end of his address to one Dionysius, though only six of them have hitherto been printed; and an imperfect book on multangular numbers, namely in a Latin translation only, by Xilander, in the year 1575, and afterwards in 1621 and 1670 in Greek and Latin by Gaspar Bachet. These books however do not contain a treatise on the elementary parts of algebra, but only collections of difficult questions relating to square and cube numbers, and other curious properties of numbers, with their solutions. And Diophantus only prefaces the books by an address to Dionysius, for whose use it was probably written, in which he just mentions certain precognita, as it were to prepare him for the problems themselves. In these remarks he shows the names and generation of the powers, the square, cube, 4th, 5th, 6th, &c, which he calls dynamis, cubus, dynamodinamis, dynamocubus, cubocubus, according to the sum or addition of the indices of the powers; and he marks these powers with the initials thus δ^v, κ^v, $\delta\delta^v$, $\delta\kappa^v$, $\kappa\kappa^v$, &c: the

unknown quantity he calls simply *αριθμος*, *numerus*, the number; and in the solutions he commonly marks it by the final thus ϛ; also he denotes the monades, or indefinite unit, by μ°. Diophantus then remarks on the multiplication and division of simple species together, showing what powers or species they produce; declares that minus (λειψις) multiplied by minus produces plus (*υπαρξιν*); but that minus multiplied by plus, produces minus; and that the mark used for minus is ⋔, namely the ψ inverted and curtailed; but he uses no mark for plus, but a word or conjunction copulative.

As to the operations, viz, of addition, subtraction, multiplication, and division of compound species, or those connected by plus and minus, Diophantus does not teach, but supposes his reader to know them. He then remarks on the preparation or simplifying the equations that are derived from the questions, which we call reduction of equations, by collecting like quantities together, adding quantities that are minus, and subtracting such as are plus, called by the moderns transposition, so as to bring the equation to simple terms, and then depressing it to a lower degree by equal division, when the powers of the unknown quantity are in every term: which preparation, or reduction of the complex equation, being now made, or reduced to what we call a final equation, Diophantus goes no further, but barely says what the root or *res ignota* is, without giving any rules for finding it, or for the resolution of equations; thereby intimating that such rules were to be found in some other work, done either by himself or others.

Of the body of the work, lib. 1 contains forty-three questions, concerning one, two, three, or four unknown numbers, having certain relations to each other, viz, concerning their sums, differences, ratios, products, squares, sums and differences of squares, &c, &c; but none of them concerning either square or cubic numbers. Lib. 2 contains thirty-six questions. The first five questions are concerning two numbers, though only one condition is given in each

question; but he supplies another by assuming the numbers in a given ratio, viz., as 2 to 1. The 6th and 7th contain each two conditions: then in the 8th question he first comes to treat of square numbers, which is this, to divide a given square number into two other squares; and the 9th is the same, but performed in a different way: the rest, to the end, are, almost all, about one, two, or three squares. Lib. 3 contains twenty-four questions concerning squares, chiefly including three or four numbers. Lib. 4 begins with cubes; the first of which is this, to divide a given number into two cubes whose sides shall have a given sum: here he has occasion to cube the two binomials $5 + n$ and $5 - n$; the manner of doing which shows that he was acquainted with the composition of the cube of a binomial; and many other places manifest the same thing. Only part of the questions in this book are concerning cubes; the rest are relating to squares. Two or three questions in this book have general solutions, and the theorems deduced are general, and for any numbers indefinitely; but all the other questions, in all the four books, are employed in finding only particular numbers. Lib. 5 is also concerning square and cube numbers, but of a more difficult kind, beginning with some that relate to numbers in geometrical progression. Lib. 6 contains twenty-six propositions, concerning right-angled triangles; such as to make their sides, areas, perimeters, &c, &c, squares or cubes, or rational numbers, &c. In some parts of this book it appears, that Diophantus was acquainted with the composition of the 4th power of the binomial root, as he sets down all the terms of it; and, from his great skill in such matters, it seems probable that he was acquainted with the composition of other higher powers, and with other parts of algebra, besides what are here treated of. At the end is part of a book, in ten propositions, concerning arithmetical progressions, and multangular or polygonal numbers. Diophantus once mentions a compound quadratic equation; but the resolution of his questions is by simple equations, and by means of only one

unknown letter or character, which he chooses or assumes so ingeniously, that all the other unknown quantities in the question are easily expressed by it, and the final equation reduced to the simplest form which it seems the question can admit of. Sometimes he substitutes for a number sought immediately, and then expresses the other numbers or conditions by it: at other times he substitutes for the sum or difference, &c, and thence derives the rest, so as always to obtain the expressions in the simplest form. Thus, if the sum of two numbers be given, he substitutes for their difference; and if the difference be given, he substitutes for their sum: in both cases he has the two numbers easily expressed by adding and subtracting the half sum and half difference; and so in other cases he uses other similar ingenious notations. In short, the chief excellence in this collection of questions, which seems to be only a set of exercises to some rules which had been given elsewhere, is the neat mode of substitution or notation; which being once made, the reduction to the final equation is easy and evident: and there he leaves the solution, only mentioning that the root or αριθμος is so much. Upon the whole, this work is treated in a very able and masterly manner, manifesting the utmost address and knowledge in the solutions, and forcing a persuasion that the author was deeply skilled in the science of algebra, to some of the most abstruse parts of which these questions or exercises relate. However, as he contrives his assumptions and notations so as to reduce all his conditions to a simple equation, or at least a simple quadratic, it does not appear what his knowledge was in the resolution of compound or affected equations.

But though Diophantus was the first author on algebra that we now know of, it was not from him, but from the Moors or Arabians that we received the knowledge of algebra in Europe, as well as that of most other sciences. And it is matter of dispute who were the first inventors of it; some ascribing the invention to the Greeks, while others say that the Arabians had it from the Persians, and these from the

Indians, as well as the arithmetical method of computing by ten characters, or digits; but the Arabians themselves say it was invented among them by one Mahomet ben Musa, or son of Moses, who it seems flourished about the 8th or 9th century. It is more probable, however, that Mahomet was not the inventor, but only a person well skilled in the art; and it is further probable, that the Arabians drew their first knowledge of it, either from the Indians, or from Diophantus and other Greek writers, as they did that of geometry and other sciences, which they improved and translated into their own language; and from them it was that Europe received these sciences, before the Greek authors were known to us, after the Moors settled in Spain, and after the Europeans began to hold communication with them, and began to travel among them to learn the sciences. And according to the testimony of Abulpharagius, in the year 969, the arithmetic of Diophantus had been translated into Arabic by Mahomet ben-yahya Buziani. But whoever were the inventors and first cultivators of algebra, it is certain that the Europeans first received the knowledge, as well as the name, from the Arabians or Moors, in consequence of the close intercourse which subsisted between them for several centuries. And it appears that the art was pretty generally known, and much cultivated, at least in Italy, if not in Spain, as well as other parts of Europe also, long before the invention of printing, as many writers upon the art are still extant in the libraries of manuscripts; and the first authors, presently after the invention of printing, speak of many former writers on this subject, from whom they learned the art.

OF THE INDIAN ALGEBRA.

Some notices have lately been obtained of the science of algebra among the Indians; and it is very probable that, through the intercourse of learned Englishmen with that country, we shall receive still more considerable information on that head.

There has long existed cause to suspect that the principles of this art came to Europe through the Arabians and Moors, as well as the Indian numeration and arithmetic; and every extension of our concerns among them, serves further to increase the probability of that opinion. For more than a century past, evidence has been received in Europe, at various times, of the existence of very learned works on astronomy among the Indians. Such notices were first imported by certain learned Frenchmen, and communicated through the Memoirs of the Academy; whence a very ingenious and learned account of such works was given in the Astronomie Indienne of the unfortunate M. Bailly. Since then, many other valuable communications have been made by several of our own learned countrymen, belonging to the Bengal Society, and other persons curious in the sciences; as Sir William Jones, Samuel Davis, Esq. Edward Strachey, Esq. and many others. Hence the strongest evidence has been obtained, that, at a period several thousand years (at least three or four), before the Christian era, the Indians must have possessed very correct astronomical observations and rules of calculation; rules that require a considerable knowledge of geometry and of trigonometry, both plane and spherical; and even accompanied with regular tables of sines and versed sines; at a time when all Europe was in a state of gross barbarity, if it was at all inhabited. See a valuable paper in the 2d volume of the Asiatic Researches, by Samuel Davis, Esq. on the Astronomical Computations of the Hindus; also two learned dissertations on the Indian Astronomy and Trigonometry, by Professor Playfair, in the Edinburgh Philosophical Transactions, Vols. II, and IV.

What we have now, however, particularly to attend to, is the algebra of that country. It has long been thought that a people, possessing so much knowledge of many other branches of mathematical science, could not well be unacquainted with algebra; and we have now received incontestible proofs of their very critical skill in that branch,

Several specimens of such works have been seen in that country, both in the native language and in Persian translations. Some of the latter are also now in the hands of S. Davis, Esq. of Hill-street, one of the Directors of the East-India Company, with a partial translation into English; and similar translations of some others have been sent to England by Mr. Edward Strachey, before-mentioned. As I have been favoured with the perusal of these, I am enabled to give some account of them.

The first of these communications, by Mr. Strachey, of the Bengal civil establishment, is a printed account of some observations on the originality, extent, and importance, of the mathematical science of the Hindoos; with extracts from Persian translations of the Leelawuttee and the Beej Gunnit; or the Beja Ganita, as it is written by Mr. Davis. These two works, Mr. Strachey informs us, were both written by Bhasker Acharij, a famous Hindoo mathematician and astronomer, who lived about the beginning of the 13th century of the Christian era; the latter of these two treatises being on algebra, with some of its applications; and the former on arithmetic, and algebra, and mensuration or practical geometry. The Beej Gunnit was translated into Persian in 1634, by Utta Ulla Rusheedee, at Agra or Dehli probably; and the Leelawuttee in 1587, by the celebrated Fyzee.

It is well known, Mr. Strachey says, that the only Persian science is Arabian, and that the Arabs had much of their mathematical knowledge from the Greeks; it is certain, however, that they had their arithmetic from the Indians, and most likely their algebra was drawn from the same source; but the time, and other circumstances respecting the introduction of these sciences among the Arabs, is unknown. It appears, however, that the first account of any Indian mathematical science among the Arabs, was of their astronomy, which was known in the reign of Al Mamoon. In later times, many Mahommedans have had access to the Hindoo books: accounts of several are in the

Ayeen Ackbery, and in D'Herbelot. Abul Fuzl gives a list of Sanscrit books, which were translated into Persian in Akbar's time; among which the Leelawuttee is the only mathematical work.

From a comparison of the algebra of the Arabians and Greeks, and that of the modern Europeans, with the Persian translations of the Beej Gunnit and Leelawuttee, it would probably appear, that the algebra of the Arabs is quite different from that of Diophantus, and not taken the one from the other: that if the Arabs did learn from the Indians, as is most probable, they did not borrow largely from them; that the Persian translations of the Beej Gunnit and Leelawuttee contain principles, which are sufficient for the solution of any propositions in the Arabian, or in the Diophantine algebra; that these translations contain propositions, which are not to be solved on any principles that could be supplied by the Arabian or the Diophantine algebra; and that the Hindoos were further advanced in some branches of this science than the modern Europeans, with all their improvements, till the middle of the eighteenth century.

On Series, with Extracts from the Leelawuttee.

In the translation of the Leelawuttee there is a chapter on combinations, and another on progressions, as in the following specimens:—

Combinations.

" To find the number of mixtures of different things."

" If it be required to add different things together, so that all the combinations, arising from their addition, may be known, this is the rule:"

" First write them all, with 1, in order; and below write 1 the last opposite the first in order. Then divide the first term of the first line by the number which is opposite to it in the second line; the quotient will be the number of com-

binations of that thing. Multiply this quotient by the 2d term of the first line, and divide the product by the number which is opposite to it in the second line, the quotient will be the number of combinations of that thing. Again, multiply this quotient by the 3d term, and divide by that which is below it, and so on. Then add together whatever is thus obtained below each term, the sum will be the amount of all the combinations of these things."

Such is their manner of expressing all their algebraic rules, in words at length, which sometimes, in complex cases, makes them difficult to follow, and easily mistaken. Whereas the same rule is soon comprehended at sight, when expressed in our own convenient mode and notation; as in the present case, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ &c, denotes the combinations of any number (n) of things, taken two by two, three by three, four by four, &c, the series being continued to as many factors as there are things to be combined. The translation then continues in the following example:—

Example. "The six-flavoured, called in Hindu, *Khut Rus*, contains, 1st a sweet, 2d a salt, 3d a sour, 4th a soft, 5th a bitter, 6th a sharp: I would know the number of different mixtures which may be had by adding these together. Write them thus:

$\left\{\begin{matrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\}$; then $\frac{6}{1} = 6$, $\frac{6 \times 5}{2} = 15$, $\frac{15 \times 4}{3} = 20$, $\frac{20 \times 3}{4} = 15$, $\frac{15 \times 2}{5} = 6$, $\frac{6 \times 1}{6} = 1$, the sum is 63. The number of mixtures then of 6 things is 63."

Progressions.

"Of numbers increasing. This may be of several kinds. First, with the number 1, that is, when each term exceeds the preceding by 1. The way to find the sum to any number is; add 1 to that number, and multiply by half that number. To find the sum of the terms arising from the continued addition of the former terms; add 2 to the num-

ber of terms, then multiply by the sum of the same number of terms of the first series, and divide by 3, the quotient will be the sum of the same number of the latter terms."

These rules are, in short, first to find the sum of any number of terms (n) in the series of natural numbers 1, 2, 3, 4, 5, 6, &c; and then to find the sum of any number of the terms arising by the continued addition of the former, being what we call the triangular numbers. Thus,

1, 2, 3, 4, 5, &c, the natural numbers.
1, 3, 6, 10, 15, &c, the triangular numbers.

Our rule for the former series is $\frac{1}{2}n \times (n+1) = s$; and for the latter series $\frac{1}{6}n \times (n+1) \times (n+2) = s \times \frac{1}{3}(n+2)$, where the latter form agrees with that given in the Indian rule.

Exam. In the first series; for four terms, $1+2+3+4 = (1+4) \times \frac{4}{2} = 10$; for six terms, $(1+6) \times \frac{6}{2} = 21$; for nine terms, $(1+9) \times \frac{9}{2} = 45$. In the second series; for three terms, $\frac{(3+2) \times s}{3} = \frac{(3+2) \times (1+2+3)}{3} = \frac{5 \times 6}{3} = 10$; for four terms, $\frac{(4+2) \times s}{3} = \frac{(4+2) \times (1+2+3+4)}{3} = \frac{6 \times 10}{3} = 20$; for nine terms, $\frac{(9+2) \times s}{3} = \frac{(9+2) \times 45}{3} = 165$.

The next rule is for the summation of series of square and cube numbers, applied to the series 1, 4, 9, 16, 25, &c, and 1, 8, 27, 64, 125, &c. The sum of n terms of the former being $\frac{2n+1}{3} \times s$, and the sum of n terms of the latter s^2, where s denotes the sum of n terms of the natural series 1, 2, 3, 4, &c. Rules are then given for any arithmetical progressions, a being the first term, m the middle term, z the last term, d the common difference, and s the sum; for which the rules, expressed in our algebraical notation, are these: $(n-1)d + a = z$; $\frac{z+a}{2} = m$; $mn = s$; $\frac{s}{n} - (n-1)\frac{d}{2} = a$; $(\frac{s}{n} - a) \div (\frac{n-1}{2}) = d$; $\frac{\sqrt{(2ds + (a-\frac{1}{2}d)^2)} - (a - \frac{1}{2}d)}{d} = n$;

where the last form is deduced from the preceding one, by means of the solution of a compound quadratic equation, of a rather complex nature, by the mode of completing the square, and evincing a considerable degree of expertness in the arrangement.

Next follows the rule for summing a geometrical progression, thus:

"A person gave a certain number the first day; the 2d day he added that number multiplied by itself; and so on for several days, adding every product multiplied by the first number. The rule for finding the sum is this: First observe whether the number of the times is odd or even; if it is odd, subtract 1 from it, and write it somewhere, placing a mark of multiplication above it; if it is even, place a mark of a square above it, and halve it till the number of the times is finished; after that, beginning at the bottom with the number of increase, multiply where there is a mark of multiplication, and take the square where there is a mark of a square; subtract 1 from the product, and divide what remains by the number of increase less 1, and multiply the quotient by the number of the first day; the product will be the sum of the progression."

"Example, where the number of terms is even, 2, 4, 8, &c, to thirty terms.—Example, where the number of terms is odd, 2, 6, 18, 54, &c, to seven terms."

In these examples, the result only is given, without any detail of the application of the rule to the case. Indeed, the rule is not very clearly expressed, though one easily perceives what is meant by it; and which comes to the same thing as our own rule, thus expressed, $\frac{r^n-1}{r-1} \times a = s$, or $\frac{a^n-1}{a-1} \times a = s$, when the ratio r and the first term a are equal.

In this rule, we perceive indications that the Hindoos had a mark to denote multiplication, and another to denote squaring. It may be doubted, whether some of the rules in these two chapters were known in Europe till after the 16th

century. We know that Peletarius, in his algebra, printed in 1558, gave a table of square and cube numbers; and remarked, among other properties of these numbers, that the sum of any number of cubes, taken from the beginning, always makes a square number, the root of which is the sum of the roots of the cubes; which is the same thing as the Leelawuttee rule before-mentioned. It does not appear whether the Hindoos had any knowledge of figurate numbers, beyond what is given above.

On the Mensuration of the Circle and Sphere, with Extracts from the Leelawuttee.

The rules in the Leelawuttee for the mensuration of the circle and the sphere are these:

To find the circumference of a circle, multiply the diameter by 3927, and divide the product by 1250; or multiply the diameter by 22, and divide by 7.

In this precept we find two approximations; one of them, 7 to 22, the same ratio as had been the result of one of the labours of the prince of the ancient mathematicians, Archimedes; and the other 1250 to 3927, the very same as 1 to 3·1416, nearer than those of any of the Europeans before the labours of Vieta.

Among other rules in mensuration, they have the following: D being the diameter, and C the circumference; then $\frac{1}{4}DC$ = area of the circle; also DC = surface of the sphere, and $\frac{1}{6}D^2C$ = its solidity.—Other rules are, $\frac{3927}{5000}D^2$ = area, which is exactly equivalent to our $\cdot 7854D^2$, and shows that it is derived from the ratio above-mentioned, 1250 to 3927. Another rule for the same is, $\frac{11}{14}D^2$ = area, which is another of our approximations, and derived from the ratio 7 to 22. —Another rule given for the solidity of the sphere in terms of the diameter, is $\frac{D}{2} + \frac{D^3}{\frac{2}{21}}$ = the solidity of the sphere, which must be wrong printed in various respects. Having endeavoured to restore this form to some probability, I have

imagined it might be intended to mean the same as $\frac{D}{2} \times \frac{D^2}{\frac{20}{21}}$, which reduces to $\frac{D}{2} \times \frac{21}{20}D^2 = \frac{21}{40}D^3 = \cdot525D^3$, and is very nearly the same as our own approximation $\cdot5236D^3$. —Again, D being the diameter, c the chord, and v the versed sine of an arch a, then $\frac{1}{2}D - \frac{1}{2}\sqrt{(D^2 - c^2)} = v$, a theorem geometrically correct: from which is derived $2\sqrt{(Dv - v^2)} = c$, which is correct also.—But the two following are only approximations, where C = the circumference, viz, $\frac{4aD(C-a)}{\frac{1}{4}C^2-(C-a)a} = c$, which gives the chord c rather too great, by about the 58th part in an arc of two degrees, and about the 166th part in an arc of thirty degrees, being true to the 4th place of figures in the former case, and to the 3d place in the latter; but it does not appear how they have got this rule. From this however they have derived the following rule, by resolving a compound quadratic equation, viz, $\frac{C}{2} - \sqrt{(\frac{C^2}{4} - \frac{\frac{1}{4}C^2c}{4D+c})} = a$.

The following are their rules for finding the sides of regular figures inscribed in a circle, for the polygons of 3, 4, 5, 6, 7, 8, 9, sides, viz, of the

Trigon	$\frac{130323}{120000}$ D.	Heptagon	$\frac{52055}{120000}$ D.	But should be	
Tetragon	$\frac{84853}{120000}$ D.	Octagon	$\frac{45922}{120000}$ D.	Trigon	$\frac{103923}{120000}$ D.
Pentagon	$\frac{70534}{120000}$ D.	Nonagon	$\frac{41031}{120000}$ D.	Heptagon	$\frac{52070}{120000}$ D.
Hexagon	$\frac{60000}{120000}$ D.			Nonagon	$\frac{41042}{120000}$ D.

Where three of the numbers are erroneous, viz, in the tetragon, the heptagon, and the nonagon, most probably by miscopyings: the correct numbers are here set in the last column.

On Quadratic Equations, with Extracts from the Leelawuttee and the Beej Gunnit.

In the translation of the Leelawuttee, the rule for quadratic equations is:—

"When the number, by which the root of the number thought of is multiplied, is given; and the sum or difference of the product when added to, or subtracted from the number thought of, is also given; the rule for finding the number is: Take half the multiplier and square it, add what remains to it; take its root; add or subtract half the multiplier to the root, according as the question is of subtraction or addition; take the square of the aggregate, and that will be the number thought of."

Now this is exactly our process for the equation $x \pm a\sqrt{x} = b$, by completing the square, &c, which gives $x = (\sqrt{(\frac{1}{4}a^2 + b)} \mp \frac{1}{2}a)^2$.

A rule is given, for clearing the higher power from fractional coefficients, thus:—

"To add to the number thought of, or to subtract from it, a fraction of that number, the rule is: Add or subtract the fraction with 1; divide the multiplier and the remainder by the sum or difference, and proceed according to the foregoing rule, with the quotients of the multiplier and remainder."

That is, in our notation, having the equation $x \pm \frac{1}{m}x \pm a\sqrt{x} = b$: first, dividing by $1 + \frac{1}{m}$, gives $x + \frac{a}{1+\frac{1}{m}}\sqrt{x} = \pm\frac{b}{1+\frac{1}{m}}$; then proceeding as before gives

$$x = [\sqrt{((\frac{\frac{1}{2}a}{1+\frac{1}{m}})^2 + \frac{b}{1\pm\frac{1}{m}})} \mp \frac{\frac{1}{2}a}{1\pm\frac{1}{m}}]^2.$$

The translations from the Beej Gunnit are to the same purport, but rather more obscurely expressed. One of the cases is for the equation $ax^2 + bx = c$, and the method given is this: multiply all by $4a$, this gives $4a^2x^2 + 4abx =$

$4ac$; next add the square of b, this gives $4a^2x^2+4abx+b^2=b^2+4ac$; the roots give $2ax+b=\sqrt{(b^2+4ac)}$; then $x=\frac{\sqrt{(b^2+4ac)}-b}{2a}$, which process, by avoiding fractions, is much easier than our own method in such cases of quadratics.

With respect to the double roots, the same translation says:

"When one side is the thing, and the numbers are negative, and on the other side the numbers are less than the negative numbers on the first side, there are two methods: The first is, to equate them without alteration; the second is, if the numbers of the second side are affirmative, to make them negative; and if negative, to make them affirmative. Equate them; two numbers will be obtained, both of which will probably answer."

This, though obscurely expressed, appears to relate to the two positive roots of certain quadratic equations. Whence we perceive that in the translations of the Bija Ganita, and the Lilawati, the solution of quadratic equations is carried as far as it was among the Arabs, and the modern Europeans before Cardan, but no further. Lucas de Burgo, whose book was printed near the end of the 15th century, used both the roots or values of the unknown quantity, in that case of the quadratics which has two positive roots; but he takes no notice of the negative roots in the other two cases.

One of the examples given of quadratics in the Bija Ganita, is this:

"Some bees were sitting on a tree; at once the square root of half their number flew away; again nine-tenths of the whole flew away, after which two bees remained: how many were there?"

Considerable address is manifested in the solution of this question, which produces a rather intricate equation when, in the common way, x is assumed for the number sought, as the equation from the question then is $x-\frac{8}{9}x-\sqrt{\frac{1}{2}x}=2$,

or $\frac{1}{9}x - \sqrt{\frac{1}{2}x} = 2$. But, instead of this way, the Hindu assumes $2x^2$ for the number of bees, which at once gives this simpler form $2x^2 - \frac{16}{9}x^2 - x = 2$, or $\frac{2}{9}x^2 - x = 2$, the solution of which is much easier.

On Indeterminate Problems of the Second Degree, Diophantus, and the Beej Gunnit, or rather Bija Ganita.

The 16th question of the 6th book of Diophantus, is as follows:—

"Having two numbers given, if one of these drawn into a certain square, and the other subtracted from the product, make a square; it is required to find another square, greater than the former, which shall do the same thing. For instance, given the two numbers 3 and 11, and a certain square 25, which drawn into 3, and 11 taken from the product, leaves the square of 8; to find another square greater than 25 having the same property. Put its side 1N + 5, then its square is 1Q + 10N + 25; triple of this diminished by 11 leaves 3Q + 30N + 64 equal a square; let its side be 2N − 8, which gives N = 62; then 62 + 5 = 67 is the side, and 4489 the square sought."

In the Bija Ganita this problem is solved very generally and scientifically, by the assistance of another, which was unknown in Europe till the middle of the 17th century; and first applied to questions of this nature by Euler, in the middle of the 18th century.—With the affirmative sign, the Bija Ganita rule for finding new values of $ax^2 + b = y^2$, is this: Suppose $ag^2 + b = h^2$ a particular case: find m and n such that $an^2 + 1 = m^2$; then is $x = mg + nh$, and $y = mh + ang$.

General methods, according to the Hindus, for the solution of indeterminate problems of the first and second degrees, are found in the 4th and 5th chapters of the Bija Ganita, which differ much from Diophantus's work. It contains, in Mr. Strachey's opinion, which is highly probable, a great deal of knowledge and skill, which the Greeks

had not; such as, the use of an indefinite number of unknown quantities, and the use of arbitrary marks to express them; a good arithmetic of surds; a perfect theory of indeterminate problems of the first degree; a very extensive and general knowledge of those of the second degree; a perfect acquaintance with quadratic equations, &c. The arrangement and manner of the two works are as different as their substance: the one constitutes a regular body of science; the other does not: the Bija Ganita is quite connected and well digested, and abounds in general rules, which suppose great learning; the rules are illustrated by examples, and the solutions are performed with skill. Diophantus, though not entirely without method, gives very few general propositions, being chiefly remarkable for the dexterity and ingenuity with which he makes assumptions for the simple solution of his questions. The former teaches algebra as a science, by treating it systematically; the latter sharpens the wit, by solving a variety of abstruse and complicated problems in an ingenious manner. The author of the Bija Ganita goes deeper into his subject, treats it more abstractedly, and more methodically, though not more acutely, than Diophantus. The former has every characteristic of an assiduous and learned compiler; the latter of a man of genius in the infancy of science.

Besides the foregoing remarks, derived chiefly from the printed notices of Mr. Strachey, I have lately been favoured with communications of several other curious particulars relating to the same two books, by S. Davis, Esq. The late Mr. Reuben Burrow collected, in India, many oriental manuscripts on the mathematical sciences, both in the Sanskrit and the Persian languages, the latter being translations only of the former: most of these he bequeathed by will to one of his sons there, but with an injunction not to be delivered to him till he should have learned those languages and the sciences. But one or two of these Burrow left to his friend Mr. Dalby, mathematical professor at the Royal Military College, Wycombe. These are now in Mr. Dalby's possession, being

the Persian translations of the Bija Ganita and Lilawati, with an attempt at an English translation of some parts of them by Mr. Burrow; but these attempts being mostly interlineations written with a black-lead pencil, are in danger of being obliterated. Mr. Dalby has also lately received from Mr. Strachey, now in India, an entire English translation of the Bija, of a part of which he has favoured me with the perusal, and besides communicated by letter many descriptive remarks of those works, from all which sources I have collected the following curious particulars.

The first work is called the Beej, or the Beej Gunnit (as they are pronounced, but written Bija Ganita), and seems to have been translated into Persian about the year 1634. Burrow calls the algebra simply the "Beej;" but the Persian introduction has it "Beej Gunnit, the author Bhasker Acharya, author of the Leelawuttee." It is there also said, that "this excellent method of computation was translated from the Hindu into Persian, and is called the Book of Composition and Resolution;" and "that it is not written in any book, Persian or Arabic." The two words here translated "composition and resolution," are elsewhere simply called "algebra." The two Persian words for "Beej Gunnit," are totally different from them; so that the Persians and Arabians have adopted the *meaning*, not the *pronunciation*, of the Sanskrit. Part of the MS. is a commentary on the original Hindu work, by the Persian translator, who in one place refers to a "Dictionary of Algebra," but without mentioning any author or date.

The work consists of five parts. It commences with explaining affirmatives and negatives, which he characterises by two terms denoting *existing* and *non-existing*, also *property* and *debt*. Then follow the first rules, as with us. Next, surds are given at great length; and there seems here a general method of finding the number of surds in any powers raised from multinomial surd roots, something like our method of finding the number of combinations. Next follow questions about

squares; such as (using our own notation), finding $67x^2+1$; and $61x^2+1$, and $13x^2-1$, &c, squares; also a general way or method to make expressions of this kind ax^2+b squares.

Then questions are treated producing simple equations; with the application to some questions about triangles. Then more questions about squares; as, to find $x \pm y$ and xy both squares; also x^3+y^3 and x^2+y^2 both squares.

Next, some questions producing quadratics. Something is also said of a cubic; but it seems to hint that it cannot be solved generally: a straight ruler is mentioned, which it is suspected may allude to some mechanical method of solution, for there is an omission in the translation. Then more about squares; such as, to find

$\left.\begin{matrix} x-y \\ x^2+y^2 \end{matrix}\right\}$ both squares; $\left.\begin{matrix} 7x^2+8y^2 \\ 7x^2-8y^2+1 \end{matrix}\right\}$ both squares; $\left.\begin{matrix} x^2+y^3 \\ x+y \end{matrix}\right\}$ squares;

$\left.\begin{matrix} x-y+2 \\ x+y+2 \end{matrix}\right\}$ squares; $\left.\begin{matrix} x^2-y^2+1 \\ x^2+y^2+1 \end{matrix}\right\}$ squares; $\left.\begin{matrix} x-y \text{ a square} \\ x^2+y^2 \text{ a cube} \end{matrix}\right\}$

in whole numbers; $\frac{x^3-a}{b}$ a whole number, &c, &c.

Lastly, more indeterminate questions. In short, a great part is about indeterminate, and what we call questions of the Diophantine kind, yet without any one being the very same as in that author; which alone seems to show a different origin; besides, they are mostly treated in a very different way. Several of them are not easy; and Mr. Burrow has sometimes given answers in the margin, done in the European manner.

There are three or four questions about right-angled triangles, done algebraically; and here one might have expected to have found Eucl. 47, I, quoted; but, instead of that, a reference is made to the "figure of wedding chair." But from these, and many other parts of their writings, we perceive that they were possessed of, not this property only, but most others the same as in our geometry; and here it was probably that Pythagoras acquired his mathema-

tical knowledge, which he carried back with him, and taught to his countrymen. The results of the operations by the cipher 0, are the same as we make them: the quotient $\frac{a}{0}$ Burrow at first translated "infinite," but afterwards crossed it with his pencil, and substituted "cannot be comprehended:" probably it means *unbounded*, or *unassignable*.

Respecting the notation in this MS. it appears that unknown quantities are represented by letters or characters, which they call "colours." Thus, when only one unknown is used, it is denoted by a character called "Majool;" when two unknowns enter into the computation, the second is "Aswad" (black); a third is "Neelok" (blue); the fourth yellow, &c. The mark for Majool is this مجهول; Aswad thus اسود; Neelok thus نيلك; the others are more simple. Affirmative is ماں; and negative وني. The mark for the unknown, or quantity sought, in an equation, is شي, and is called technically the *thing*, meaning emphatically the *thing* about which the question or enquiry is made. All these marks, denoting words expressing the operations, are Arabian. And it is remarkable that the first Italian writers on algebra, used a word of the same import for the unknown quantity, viz, *cosa*, the *thing*; whence, in Europe, the science came to be called the *cossic art*, and such quantities *cossic numbers*, &c.

There is also a character, viz, مال, set next an unknown, to denote its square; another, كعب, for the cube, &c: also a mark for the square root; and one for the cube root: but there does not appear to be any thing answering to a vinculum used in compound quantities. The higher powers are formed, and named, by repeating and combining these marks; so, the 4th power is called the square square; the 5th power, the square cube; the 6th power, the cube cube; and so on, meaning those powers multiplied together. For

equality they use the expression "I equalize," or "equalizing," and, indeed, all their operations are expressed in words at length.

Mr. Davis however informs me that, instead of most of the preceding words in their notation, the Indians commonly contract them, so as to employ only the initial letter of the words, as was also the custom of the Europeans for a long time afterwards in their writings. Those initials the Hindus employ as we now do the ordinary letters for numbers, a, b, c, &c. Their multiplication is represented by joining them together as we do, abc. To denote their addition, the mark $\|$ is set between them: so $a+b+x$ is $a \| b \| x$. If any quantity is negative or subtractive, a point is set over it; as $a+b-x$ is $a \| b \| \dot{x}$. So also $ab\dot{x}$ denotes $ab \times -x$. They have also a method of denoting the powers and roots of a quantity by subjoining the initial of the name of the power or root to the quantity. So that if s should be the initial of the square, then as would denote aa or a^2; and ar would denote $\sqrt{a}$, if r were the initial of their word for root: thus using the mark after the quantity, as we now use it before the same.

Mr. Davis further informs me that the Sanskrit characters or letters, by which the Hindus denote the unknown quantities in their notation, are the following: पा, का, नी, पी, लो. And Charles Wilkins, Esq. the oriental librarian to the honourable East India Company, has obligingly furnished me, not only with types to print these and the Indian numeral figures with, but also the explanation of the above, which is as follows. The first, *pā*, is the initial or contraction of *pāndŭ*, pale or white; the second, *kā*, the initial of *kālă*, black; the third, *nī*, the initial of *nīlă*, blue; the fourth, *pī*, the initial of *pīlă*, yellow; and the fifth, *lō*, the initial of *lōhĭtă*, red.

Respecting the etymology of the Indian names of the books, Mr. Wilkins also expresses his opinion as follows.

Līlāvătī is an adjective in the feminine gender, formed by a particular affix from *Līlā*, sport, play, amusement, wantonness, endeavour, research, and, as the epithet of a work, may be interpreted the Book of Amusement, or the Book of Research.—*Bījă*, properly *Vījă*, signifies a seed, and thence the source whence any thing springs.—*Gănĭtă*, is the perfect participle of the root *gan*, to count, reckon, number, compute, calculate, and is interpreted by counted, reckoned, numbered, computed, calculated.—*Bījă gănĭtă* then is a compound epithet literally signifying *the seed counted, the source or root calculated.*—N. B. The names of books in Sanskrit are seldom descriptive of their contents.

It is observable too that the affirmative, or the negative sign, and the coefficient, are always placed on contrary sides of the unknown, not $+2xy$ as with us, but $2xy+$, or $+xy2$; this latter way is in the Persian, as they read from right to left; but the Hindus from left to right, like the Europeans. But, though the Persians write from right to left, they do not use, or translate the Sanskrit numerals backwards; for example, the MS. on algebra is 153 leaves, and they are regularly numbered up to ۱۵۳ (153): not with Hindu figures, but Persian; but confessedly borrowed from the Hindus, though they now differ from them in shape; a circumstance which renders the opinion very probable, that the Persians derived their calculation from the Indians, but probably at second hand through the Arabians.

Compounds are multiplied as we fill the multiplication table: for example, to multiply $-3x+2y-2$ by $+4y-x+1$, they proceed as in the margin, and afterwards collect the products.

	$-3x$	$+2y$	-2
$+4y$	$-12xy$	$+8yy$	$-8y$
$-x$	$+3x^2$	$-2xy$	$+2x$
$+1$	$-3x$	$+2y$	-2

In the process of their solutions, there is no distinct arrangement of the different steps, as in the modern algebra;

but it is carried on without any breaks in the lines, something like de Burgo, or Bombelli, &c. The language is written in the first person, as,—I do so and so—I then equalize—next I multiply, &c.—We distinguish the beginning of chapters, &c. only by being in red ink.

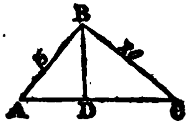
We have already noticed their method of resolving compound quadratic equations, the same as ours, by completing the square; and better than ours in such cases as $ax^2+bx=c$, when the first term has a coefficient a, which does not divide the two b and c, by which they avoid the operation in fractions. In imitation of this, also, I see they have a method of completing the powers in some cases of cubics and biquadratics: Thus, having the cubic equation $x^3 + 12x = 6x^2 + 35$; first subtracting $6x^2$, gives $x^3 - 6x^2 + 12x = 35$; next subtracting 8, gives $x^3 - 6x^2 + 12x - 8 = 27$, which completes the binomial cubic, and the roots are $x - 2 = 3$, or $x = 5$.—Again, having given the imperfect biquadratic $x^4 - 400x - 2x^2 = 9999$, a case which it is not very obvious how to bring it to a complete power, but which is managed with much address, in this manner. First add $400x + 1$ to both sides, this gives $x^4 - 2x^2 + 1 = 10000 + 400x$, where the first side is a complete square, and the roots are $x^2 - 1 = \sqrt{(10000 + 400x)}$; but as the latter side is not a complete square, the author goes back again, and tries another course; thus, to the original equation he adds $4x^2 + 400x + 1$, which gives $x^4 + 2x^2 + 1 = 4x^2 + 400x + 10000$, two complete squares, the roots of which are $x^2 + 1 = 2x + 100$; again, subtract $2x$, and it becomes $x^2 - 2x + 1 = 100$, which are again two complete squares, the roots of which are $x - 1 = 10$, and hence $x = 11$. And this process has some resemblance to that which was afterwards practised, if not imitated, by Lewis Ferrari. It appears, however, that the Indians had no general method for all equations of these two powers, but only depended on their own ingenuity for artfully managing some particular cases of them; for, at the conclusion of the above process, the author emphatically adds, "The solution of such ques-

tions as these depends on correct judgment, aided by the assistance of God."

At the conclusion of the first book occur some curious circumstances in the solution of certain problems in the application of algebra to geometry, particularly some relating to right-angled triangles, by which it appears that they were well acquainted with the most remarkable properties in Euclid's Elements, some of which are cited under names peculiar to themselves, and sometimes, Mr. Strachey thinks, by numbers of the books and propositions in Euclid's collection, unless these last references have been added by the translator, or some transcriber, which is to be suspected. Mr. Strachey says, in two examples the Elements are refered to, though not by name, viz, the 4th and 8th props. of the 2d book; the 4th expressly in these terms, " by the 4 fig. of the 2d book;" and the 8th in these words, " by this figure;" and a fig. for the demonstration of that prop. is in the margin. The 47th of the 1st book is cited under the designation of " the figure of the bride;" and in one of the examples are these words, " for the sum of the sides is always greater than the hypothenuse, by the ass's proposition."

One of these problems is this: Given the two sides about the right angle, of a right-angled triangle, 15 and 20; required the hypothenuse. To the solution it is observed that, " Although, by the *figure of the bride*, the hypothenuse is the root of the sum of the squares of the two sides, the method of solution by algebra is this." The triangle is supposed to be divided into two other triangles, by a perp. from the point of the right angle to the hypothenuse. " Then by four proportionals I find, when the least side about the right angle, whose hypothenuse is 1 unknown, is 15, how many will be the least side about the right angle, whose hypothenuse is 15." In like manner the other segment of the hypothenuse is found. Whence $\frac{625}{x} = x$, and $x = 25$. The perp. is found from the segments of the hypothenuse, and the sides which are contiguous to them.

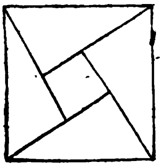
The process above mentioned, in the original, is rather obscure, but the meaning is this: Putting the hypothenuse AC $= x$, then by similar triangles, $x : 15 :: 15 : \frac{225}{x} =$ AD; in like manner $x : 20 :: 20 : \frac{400}{x} =$ CD; the sum of these two is $\frac{625}{x} =$ AC $= x$; then $625 = x^2$, and $x = \sqrt{625} = 25$.



And this would be a ready way of proving the property of the right-angled triangle, taking for known or granted the proportionality of the like sides of equiangular triangles. For, putting AB $= a$, and BC $= b$, AC being x as before; then $x : a :: a :$ AD $= \frac{a^2}{x}$; and $x : b :: b : \frac{b^2}{x} =$ CD; the sum $\frac{a^2+b^2}{x} = x$, or $a^2 + b^2 = x^2$.

Next is given, in the original, this other rule: "The square of the hypothenuse of every right-angled triangle, is equal to twice the rectangle of the two sides containing the right angle, with the square of the difference of those sides. As the joining of the four triangles above-mentioned is in such a manner, that from the hypothenuse of each the side of the square will be formed, and in the middle of it there will be a square, the quantity of whose sides is equal to the difference of the two sides about the right angle of the triangle; and as the area of every right-angled triangle is half the rectangle of the two sides about the right angle, twice the rectangle of those sides, viz, 600, is equal to all the four triangles; and when I add 25, the small square, it will be equal to the whole square of the hypothenuse, that is 625, which is equal to the square of the *thing*. In many cases an effable root cannot be found, then it will be a surd. And if we do not suppose the *thing*, add twice the rectangle of the sides to the square of their difference, and take the root of the sum, it will be the quantity of the hypothenuse: And from this it is known, that if twice the rectangle of two numbers be added to the square of their difference, the re-

sult will be equal to the sum of the squares of those two numbers."



In the margin of the original, as here annexed, is drawn a figure of four equal right-angled triangles, joined in the manner above described, exhibiting a new and obvious proof of the 47th of Eucl. 1; for here are the four equal right-angled triangles, which are equal to twice the rectangle of their two perpendicular sides; and which, together with the small square in the middle, being the square of the difference of those sides, make up the large square on the hypothenuse. But, by Eucl. VII, 2, double the rectangle of two lines, with the square of their difference, is equal to the sum of their squares; therefore the square on the hypothenuse, is equal to the squares on the other two sides. And this may be considered as the Indian demonstration of the celebrated property of the sides of right-angled triangles, demonstrated in the 47th of Eucl. 1; a property so much employed by their geometricians, and so often referred to in their writings, by the names of "the figure of the bride," and "the figure of the bride's chair," and "the figure of the wedding chair;" epithets which we may conjecture have been suggested by the above figure bearing some resemblance to a palanquin, or sedan chair, in which it is the usual practice, in that country, for the bride to be carried home to her husband's house.

In the 3d book the author comes to treat of questions and equations having several unknown quantities in them. These he directs to exterminate one after another, much after our modern way, till they are reduced to one unknown only. The number of independent equations are stated to be equal to the number of the unknown quantities; but when there are not so many equations, the defect is supplied by assuming values for as many of the unknown quantities, &c, as with us. The unknown quantities are repre-

sented and called by so many different characters and names, as is our own practice also. We denote them usually by the letters x, y, z, &c; the Hindus by different colours, or letters, or other marks also. "Thus," says our author, "suppose the 1st unknown, and the 2d black, and the 3d blue, and the 4th yellow, and the 5th red, and the 6th green, and the 7th parti-coloured, and so on, giving whatever names you please to the unknown quantities which you wish to discover; and if, instead of these colours, other names are supposed, such as letters, and the like, it may be done. For what is required, is to find out the unknown quantities, and the object in giving names, is that you may distinguish the things required."

In this 3d book occur many examples of making certain analytical expressions to be squares and cubes, &c. These are mostly very curious, many of them difficult, and often involving several unknown quantities; they are generally solved in a masterly way, and that very different from the manner of Diophantus.

The two other books are on similar indeterminate problems, but gradually more and more complex and difficult. Connected with the subject of the 5th chapter of the introduction, are the following specimens of questions, which are solved in the Bija Ganita, stating them as in our notation.

1. $\frac{w}{u} + \frac{x}{u} + \frac{y}{u} + \frac{z}{u} = \frac{w^2}{u^2} + \frac{x^2}{u^2} + \frac{y^2}{u^2} + \frac{z^2}{u^2}$, and
$\frac{r^2}{a^2} + \frac{s^2}{a^2} + \frac{t^2}{a^2} + \frac{v^2}{a^2} = \frac{r^3}{a^3} + \frac{s^3}{a^3} + \frac{t^3}{a^3} + \frac{v^3}{a^3}$.

2. To find a right-angled triangle, the area of which is equal to its hypothenuse; and to find a right-angled triangle, the area of which is equal to the product of its three sides.

3. $x + y = \square$, $x - y = \square$, and $xy =$ a cube.

4. $x^3 + y^3 = \square$, and $x^2 + y^2 =$ a cube.

5. $w + 2 = \square = a^2$; $x + 2 = \square = b^2$; $y + 2 = \square = c^2$; $z + 2 = \square = d^2$; the roots a, b, c, d, of these squares, being in arithmetical progression; and

$wx + 18 = \square = p^2$; $xy + 18 = \square = q^2$; $yz + 18 = \square = r^2$; also $a + b + c + d + p + q + r = \square = 13^2$.

6. $7x^2 + 8y^2 = \square$, and $7x^2 - 8y^2 + 1 = \square$.
7. $x^2 + y^3 = \square$, and $x + y = \square$.
8. $x^2 + y^2 + xy = \square = r^2$, and $(x + y)r + 1 = \square$.
9. $\frac{1}{2}xy + \frac{1}{2}y =$ a cube $= r^3$, and $x^2 + y^2 = \square = s^2$, and $x + y + 2 = \square = t^2$, and $x - y + 2 = \square = v^2$, and $x^2 - y^2 + 8 = \square = u^2$, also $r + s + t + v + u = \square$.
10. $x + y + 3 = \square = r^2$, and $x - y + 3 = \square = s^2$, and $x^2 + y^2 - 4 = \square = t^2$, and $x^2 - y^2 + 12 = \square = v^2$, and $\frac{1}{2}xy + y =$ cube $= u^3$, also $r + s + t + v + u + 2 = \square$.
11. $x^2 - y^2 + 1 = \square$, and $x^2 + y^2 + 1 = \square$.
12. $x^2 - y^2 - 1 = \square$, and $x^2 + y^2 - 1 = \square$.
13. $3x + 1 = \square$, and $5x + 1 = \square$.
14. $3x + 1 =$ cube $= r^3$, and $3r^2 + 1 = \square$.
15. $2(x^2 - y^2) + 3 = \square$, and $3(x^2 - y^2) + 3 = \square$.
16. $\frac{x^2-4}{7} = y$ a whole number.

A rule is given for making rational $(ax + my)^2 + ry^2$, by making $ax + my = \frac{r-y}{2}$.

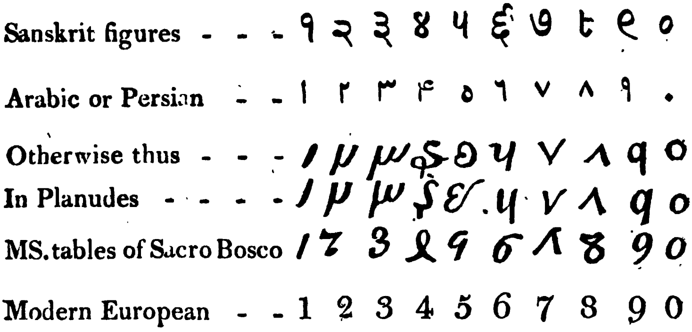
Besides the above, there are many curious things in the Bija Ganita, and it may be presumed, adds Mr. Strachey, that the Hindu books contain much interesting matter respecting this branch of algebra; it is probable that they have some system of continued-fractions; and perhaps methods of approximation, and theories of series and equations. From the rules of the 5th book, and their application, there is some ground for an opinion, that the Hindus may have had a knowledge of curves, and much of the application of mathematics to natural philosophy.

The origin of algebra, as well as of arithmetic, was probably Indian. We have, however, as yet but little certain information on this subject. It has been said that the Arabs ascribe the invention of algebra to the Greeks; but the algebra of Diophantus is widely different from that of the Arabs; and it is very doubtful whether the Greeks ever had

any other than that of Diophantus. If there be any doubt of Diophantus's algebra being of Greek origin, it may be worthy of remark, that at Alexandria he might have had the means of access to Indian literature. The Bija Ganita is, indeed, of comparatively modern date; but we must not forget that it is extracted from other books, as stated by Mr. Davis, in the Asiatic Researches, in a very learned article on the cycle of 60 years. And it is not unlikely that there are old Hindu treatises, from which not only the Bija Ganita, but even the algebra of Diophantus, and that of the Arabs, may have been derived.

The other work, the Lilawati, as before-mentioned, is on arithmetic, mensuration, &c. In the introduction, we find that "the collector of the book Lilawati was Bhasker Acharya, of Bidder city," on the northern confines of Hindustan. And "though the date of it is not known, yet in another book, made in the year 1105 of Salbahan, the reason of making the Lilawati is given." In the Ayeen Akbery (a Persian work on the manners, history, laws, &c, of the Hindus, translated by Gladwin), we find "that Acharya among the followers of Jine, is one who explains any difficulties that may occur to noviciates;" and therefore we may conclude that Bhasker was an instructor in mathematics. Now, the Salbahan, according to the Hindu chronology, commenced Anno Dom. 80; hence this book must have been written in 1185. But, all that can be inferred from this is, that in 1185 they did not know when Bhasker lived.

The Lilawati begins with the first rules of arithmetic, and goes through fractions, the extraction of roots, &c, with a good deal of what we should call alligation. Like us, they mark every 3d figure from the place of units, in extracting the cube root. A part towards the end is upon "forms," somewhat similar it seems to our permutations. The numeral figures have been gradually varied from the original of the Hindus as in the following specimens:



From which it appears that our present figures have been derived from the first, by gradual and successive alterations in the shape, by different transcribers. And hence it appears also, that these figures, as well as their use, and the arts of arithmetic, algebra, &c, have come to us from India, in their progress through Persia, Arabia, Africa, Spain, Italy, &c; and that we ought rather to call them Indian figures, &c, than Arabian.

In the operation of multiplication, they begin with the left hand figures of the multiplier, &c, as in the annexed example, to multiply ١٢ by ١٣٥, or 12 by 135, setting the successive lines of products each one place more towards the right hand.

١٢	12
١٣٥	135
١٢	12
٣٦	36
٦٠	60
١٦٢٠	1626

On a review of the whole premises it appears, that the Indians, from very ancient times, possessed all the knowledge in algebra to be found, not only in Diophantus, but in the works of the Italians, &c, before the improvements made in the time of Tartalia and Cardan, &c, and that even in a more perfect manner.

Having chiefly extracted the preceding imperfect account from the translations and commentaries of Mr. Strachey, it is greatly to be wished that this learned gentleman will con-

tinue and complete the work he has so commendably begun,* for the able performance of which he appears to be so eminently qualified, and give it to the world in a more complete and perfect form.

P. S. Since the foregoing account of the Indian Algebra was printed, I have been favoured by Mr. Strachey with the following translation of the Persian translator's preface to the Lilawati; which being at once very curious, and containing some useful particulars, is given below as a postscript to that account.

Translation of Fyzi's preface to the Lilawati.

"By order of king Akber, Fyzi translates into Persian, from the Indian language, the book Lilawati, so famous for the rare and wonderful arts of calculation and mensuration. He (Fyzi) begs leave to mention, that the compiler of this book was Bhascara Acharya, whose birth place, and the abode of his ancestors, was the city of Biddur, in the country of Deccan. Though the date of compiling this work is not mentioned, yet it may be nearly known from the circumstance, that the author made another book on the Construction of Almanacks, called Kurrun Kuttohul, in which the date of compiling it is mentioned to be 1105 years from the date of the Salibahn, an era famous in India. From that year, to this, which is the 32nd Jlahi year, corresponding with the Hejira year 995, there have passed 373 years." (Answering nearly to 1585 of the Christian era).

"It is said that the composing the Lilawati was occasioned by the following circumstance. Lilawati was the name of the author's (Bhascara's) daughter, concerning whom it appeared, from the qualities of the Ascendant at her birth, that she was destined to pass her life unmarried, and to remain without children. The father ascertained a lucky

* This it appears might best be done, by publishing a complete translation of the Bija Ganita and the Lilawati, with observations in notes at the foot of the pages.

hour for contracting her in marriage, that she might be firmly connected, and have children. It is said that when that hour approached, he brought his daughter and his intended son near him. He left the hour cup on the vessel of water, and kept in attendance a time-knowing astrologer, in order that when the cup should subside in the water, those two precious jewels should be united. But, as the intended arrangement was not according to destiny, it happened that the girl, from a curiosity natural to children, looked into the cup, to observe the water coming in at the hole; when by chance a pearl separated from her bridal dress, fell into the cup, and, rolling down to the hole, stopped the influx of the water. So the astrologer waited in expectation of the promised hour. When the operation of the cup had thus been delayed beyond all moderate time, the father was in consternation, and examining, he found that a small pearl had stopped the course of the water, and that the long-expected hour was passed. In short, the father, thus disappointed, said to his unfortunate daughter, I will write a book of your name, which shall remain to the latest times—for a good name is a second life, and the ground-work of eternal existence."—Fyzi's preface then proceeds.

"The arrangement of this translation was made with the assistance of men learned in this science, particularly the astrologers of the Deccan. Some Indian words, for which corresponding expressions were not to be found in books of this science, have been retained, as they were in the Indian language. The work has been divided into, an Introduction, some Rules, and a Conclusion."

Introduction. In this part are given explanations of some expressions in the science of calculation; with the meaning of certain terms employed in operations of the art of numbers; the divisions and terms of money, weights, measures, time, and mensurations; some of which are curious, as bearing some analogy to our own, of which they might possibly be the origin. The grain, or barleycorn, is

given by the author, as the element of weights as well as measures. Thus, two barleycorns make 1 Soorkh. The breadth of 8 barleycorns is the quantity of 1 finger; 24 fingers make the Arm or cubit; and 10 cubits make a Bamboo or rod, nearly the same as our own. The element of time is given thus: The time in which a word of two letters, as *Ta* or *Ka*, can be uttered 10 times, "neither slowly nor quickly," is called Pran; then 6 prans make 1 Pul; 60 puls 1 Ghurry; and 60 Ghurrys one day and night. So that it hence appears that,

60	ghurrys are	= our	24	hours
1	ghurry -	= -	24	minutes
1	pul -	= -	24	seconds
1	pran -	= -	4	seconds
1	ka uttered in		$\frac{2}{5}$	of a second.

*** In page 176, line 20, for 1626 read 1620.

ON THE ARABIAN ALGEBRA.

Since the foregoing account of the Indian algebra was written, and indeed set up by the printer, I have been favoured by Mr. Davis with the perusal of a valuable paper on the algebra of the Arabians, written by Mr. Strachey, composed and printed in India, for the Asiatic Researches, which I shall here take the liberty to give an account of, with some extracts. After some pages of introduction, Mr. Strachey says,

"The Greek algebra may be seen in Diophantus, who is the only Greek writer on the subject that we have heard of. —The Indian algebra may be seen in Bija Ganita, and in the Lilawati, by the author of the Bija: and as the Persian translations of these works contain a degree of knowledge, which did not exist in any of the ordinary sources of science, extant in the time of the translators, they may be safely taken as Indian, and of ancient origin. To give some idea of the algebra of the Arabians, by which we may

be enabled to judge whether, on the one hand, it could have been derived from Diophantus, or, on the other, that of the Hindus could have been taken from them, the work entitled Khulasat-ul-Hisab, may be taken as a specimen; especially because there is a part of this book which marks the limits of algebraical knowledge, in the time of the writer.

"We have seen that the first European Algebraists learnt of the Arabians; but no account has been given of the nature, the extent, and the origin of Arabian algebra. No distinct abstract or translation of any Arabic book, on the subject, has appeared in print; nor has it been established beyond controversy, who taught the Arabians. The Khulasat-ul-Hisab is of considerable repute in India: it is thought to be the best treatise on algebra, and it is almost the only book on the subject read here. I selected it, because I understood that, as well as the shortest, it was the best treatise that could be procured. Besides general report, I was guided by the authority of Maulavi Roshen Ali, an acknowledged good judge of such matters, who assured me that, among the learned Moslems, it was considered as a most complete work, and that he knew of no Arabian algebra beyond what it contained. In the Sulafat-ul-Asr, a book of biography, by Nizam-ul-din-Ahmed, there is this account of Baha-ul-din, the author of the Khulasat-ul-Hisab. 'He was born at Balbec, in the month of D'hi lhaj, * 953 Hijri, and died at Isfahan in Shawal, † 1031.' Mention is made of many writings of Baha-ul-din, on religion, law, grammar, &c, a treatise on astronomy, and one on the astrolabe. In this list of his works, no notice is taken of his great treatise on algebra, the Behr-ul-Hisab, which is alluded to in the Khulasat-ul-Hisab. Maulavi Roshen Ali tells me the commentators say, it is not extant. There is no reason to believe that the Arabians ever knew more than appears in Baha-ul-din's book, for their learning was at its height long before his time.

* Anno Chr. 1575. † Anno Chr. 1653.

"From what has been stated it will appear, that from the Khalasat-ul-Hisab, an adequate conception may be formed of the nature and extent of the algebraical knowledge of the Arabians; and hence I am induced to hope, that a short analysis of its contents will not be unacceptable to the society. I deem it necessary here to state that, possessing nothing more than the knowledge of a few words in Arabic, I made the translation, from which the following summary is abstracted, from the *viva voce* interpretation into Persian of Maulavi Roshen Ali, who perfectly understood the subject and both languages, and afterwards collated it with a Persian translation, which was made about * 60 years after Baha-ul-din's death, and which Roshen Ali allowed to be perfectly correct.

"The work, as stated by the author in his preface, consists of an introduction, 10 books, and a conclusion. The introduction contains definitions of arithmetic, of number, which is its object, and of various classes of numbers. The author distinctly ascribes to the Indian sages the invention of the nine figures, to express the numbers from 1 to 9. Book 1 comprises the arithmetic of integers. The rules enumerated under this head are, Addition, Duplation, Subtraction, Halving, Multiplication, Division, and the Extraction of the Square Root. The method of proving the operation, by casting out the 9's, is described under each of these rules. The author gives the following remarkable definitions of multiplication and division: viz. 'Multiplication is finding a number such, that the ratio which one of the factors bears to it, shall be the same which unity bears to the other factor; and division is finding a number, which has the same ratio to unity, as the dividend has to the divisor.'

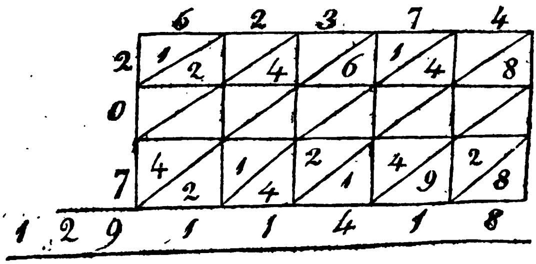
"For the multiplication of even tens, hundreds, &c, into one another, the author delivers the following rule, which is remarkable in this respect, that it exhibits an application of something resembling the indices of logarithms." 'Take

* About anno Christi 1718.

the numbers as if they were units, and multiply them together, writing down the product. Then add the numbers of the ranks together, the place of units being 1, that of tens 2, &c; subtract 1 from the sum, and call the remainder the number of the rank of the product.' This is similar to our own practice in such cases, when we say, Multiply the significant figures together, and subjoin to their product all the ciphers that are in both factors."

There next follow several other ingenious contrivances, and compendiums, such as, To multiply numbers between 5 and 10. To multiply units into numbers between units and 20. To multiply together numbers between 10 and 20. To multiply numbers between 10 and 20 into compound numbers between 20 and 100. To multiply numbers between 20 and 100, where the digits in the place of tens are the same. To multiply numbers between 10 and 100, when the digits in the place of tens are different. To multiply two unequal numbers, whose sum is even; from the square of the half sum subtract the square of the half difference. For multiplying numbers consisting each of several places of figures, the method described by this author, under the name of Shabacab, or network, and illustrated by the following example, has some resemblance to the operation by Napier's bones.

Multiply 62374 by 207.



Where, instead of our way, the method is, to set down the whole of each product in the alternate checquers, then add up the columns diagonal wise. On the other rules, nothing is

delivered differing so much from those contained in our common books of arithmetic, as to require particular mention.

Book 2d, contains the arithmetic of fractions. Book 3d, the rule of three, or to find an unknown number by four proportionals. Book 4th delivers the rule of position, both single and double, or to find an unknown number by assuming a number once or twice, and comparing the errors. Book 5th gives the method of finding an unknown number, by reversing all the steps of the process described in the question. This last rule has a near affinity to the reduction of equations in algebra, and might very naturally lead to the invention of algebra itself, being easily and generally applicable to the solution of questions usually given in the rules of position, and in the reduction of equations. For instance, taking the first example in Double Position, in vol. 1, of my Course of Mathematics: viz, What number is that, which being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient shall be 20.

Now, beginning with the last result 20, and performing in a retrograde order all the reverse operations, they will be thus: because the last operation was *dividing* by 9, to give the result 20; and multiplication being the reverse of division, therefore *multiply* the result 20 by the 9, and it gives 180; then, because the next preceding operation was *adding* 18, therefore *subtracting* 18 from the 180, leaves 162; lastly, because the next preceding operation was *multiplying* by 6, therefore *divide* the 162 by 6, and it gives 27, which must be the number sought, or that first begun with. The proof is thus: $27 \times 6 = 162$, then $162 + 18 = 180$, and $180 \div 9 = 20$. Or thus, expressing all the operations by their respective marks or characters, it will give $\frac{27 \times 6 + 18}{9}$; now first multiplying by the divisor 9, gives $27 \times 6 + 18$; then subtracting the increase 18, leaves 27×6; and lastly, dividing by the divisor 6, must give the first or number required.

And this natural process is exactly the same as the gene-

ral rule laid down at the beginning of Simple Equations in the same volume, as the most general and natural method for the reduction of all equations. Thus, assuming x for the number sought, in the same example: then, by the question, $\frac{6x + 18}{9} = 20$;

multiply by 9, and it becomes $6x + 18 = 180$;
subtract 18, and it is $6x = 162$;
lastly, divide by 6, and it gives $x = 27$.

So that the mode of operation is exactly the same in all these ways.

BOOK vith treats of Mensuration. The introduction contains geometrical definitions. *Chap.* 1, treats on the mensuration of rectilinear surfaces. Under this head, the two following articles are deserving of notice. 1st, To find the point in the base of a triangle, where it will be cut by a perpendicular, let fall from the opposite angle. Call the greatest side the base; multiply the sum of the two less sides by their difference; divide the product by the base, and subtract the quotient from the base; half the remainder will show the point on the base, where the perpendicular falls towards the least side. This property will easily be recognized as similar to that which we usually employ in resolving that case in plane trigonometry when the three sides are given, probably borrowed from the orientals; it is also the same property as that in the 35th theorem of my Geometry.

2nd, To find the area of an equilateral triangle. Multiply the square of a quarter of the square of one of the sides by 3; then the square root of the product is the area. That is, the side being a, its square is a^2, the quarter of this is $\frac{1}{4}a^2$, the square of which is $\frac{1}{16}a^4$, multiplied by 3, it is $\frac{3}{16}a^4$, its root $\sqrt{\frac{3}{16}a^4}$ is the area. This is naturally derived from the property of rightangled triangles, and, by extracting the roots, soon reduces to $\frac{1}{4}a^2\sqrt{3}$, the same as our own rule for the same purpose.

Chap. 2nd, treats on the mensuration of curvilinear sur-

faces. We here find the same rule for the area of the circle, as one of those employed by ourselves, for the same purpose, viz, multiply the square of the diameter by 11, and divide the product by 14.

Chap. 3d, on the mensuration of solids, contains nothing sufficiently remarkable to merit particular notice. This chapter concludes with the following sentence: 'The demonstrations of all these rules are contained in my greater work, entitled Bahr-ul-Hisab (the ocean of calculations), may God grant me grace to finish it.'

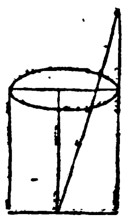
Book the viith treats on Practical Geometry. Of which, the first chapter is on levelling, for the purpose of making canals. In this are described the plummet-level, and the water-level, on the same principle with our spirit-level.

The 2nd chapter is on the mensuration of heights, accessible and inaccessible. Under the former of these heads are delivered the common methods, by bringing the top of a pole (whose height is known) in a line between the eye and the top of the height required; or by viewing the image of the top in a horizontal mirror; also by taking the proportion between a perpendicular stick of known length, and its shadow; or by taking the length of the shadow of the height when the sun's altitude is 45 degrees. The concluding method is this: 'Place the index of the astrolabe at the mark of 45°, and stand at a place where the height of the object is seen through the sights;' then measure from the place where you stand, to the place where a stone would fall from the top; add your own height, and the sum is the quantity required.

For the mensuration of inaccessible heights, the following rule is delivered: "Observe the top of the object through the sights, and mark on what shadow-line (division) the lower end of the index falls. Then move the index a step forward or backward, and advance or recede till you see the top of the object again. Measure the distance between your stations, and multiply by 7 if the index is moved

a Dhil-Kadam, and by 12 if it is moved a Dhil-Asba*, according to the shadow lines on the astrolabe; this is the quantity required.

The 3d chapter is on measuring the breadth of rivers and depth of wells. 1st. Stand on the bank of the river, and through the two sights look at the opposite bank; then turn round and look at any thing on the land side, keeping the astrolabe even: the distance from the observer to the place is the same as the breadth of the river.



2nd. Place something over the well, which shall serve for its diameter; from the centre of this diameter drop something heavy and shining, till it reach the bottom, and make a mark at the centre; then look at the heavy body through the sights of the astrolabe, so that the line of vision may cut the diameter. Multiply the distance from the mark on the diameter to the place where the line of vision cuts it, by your own height, and divide the product by the distance from the place where the line is cut to the place where you stand: the quotient is the depth of the well. This operation is obvious from a comparison with the annexed figure.

Book viii is on finding unknown quantities by algebra. In this book are two chapters. The first is introductory, chiefly relating to the formation and operation of powers of the unknown quantity, thus. Call the unknown quantity Shai (thing), its product into itself Mal (possession), the product of Mal into Shai, Cab (a die or cube), of Shai into

* In a note, Mr. Strachey says, "This part of the astrolabe consists of two squares put together laterally; the index of the instrument being at the point of the adjacent angles above. One square has 7 divisions, and the other 12; the former called Dhil-i-Kadam, the latter Dhil-i-Asba. The squares are graduated on the outer sides from the top, and at the bottom from the point of the adjacent angles. The divisions on the upright sides are those lines which Chaucer, in his treatise on the astrolabe, calls Umbra-recta; those on the horizontal he calls Umbra-versa. Chaucer's astrolabe had only one square, Dhil-i-Asba, being divided into 12 parts. The Umbra-recta is called Dhil-Mustawi, and the Versa, Dhil-Macus."

Cab, Mal-Mal; of Shai into Mal-i-Mal, Mal-Cab; Shai into Mal-i-Cab, Cab-i-Cab; and so on, without end. For one Cab write two Mals, and of these two Mals one becomes Cab; afterwards both Mals become Cab. Thus the 7th power is Mal-i-Mal-i-Cab, and the 8th Mal-i-Cab-i-Cab, in the 9th Cab-i-Cab-Cab, and so on. All these powers are in continued proportion, either ascending or descending. Thus, the ratio of Mal-i-Mal to Cab is like (or similar or equal to) the ratio of Cab to Mal, of Mal to Shai, and of Shai to 1, and of 1 to 1 divided by Shai, and of 1 divided by Shai to 1 divided by Mal, and of 1 divided by Mal to 1 divided by Cab, and of 1 divided by Cab to 1 divided by Mal-i-Mal. All this means, that all the terms in the successive powers are in continued proportion, viz, in our notation, $\frac{1}{x^4}$, $\frac{1}{x^3}$, $\frac{1}{x^2}$, $\frac{1}{x}$, 1, x, x^2, x^3, x^4, &c,

or x^{-4}, x^{-3}, x^{-2}, x^{-1}, x^0, x^1, x^2, x^3, x^4, &c.

To multiply one of these powers by another. If they are both on the same side, (viz, of unity) add the exponents of their powers together; the product will have the same denomination as this sum. For example, to multiply Mal-i-Cab by Mal-i-Mal-i-Cab, the first is the 5th power, and the other is the 7th; the result then is Cab-i-Cab-i-Cab-i-Cab, or four Cabs, which is the 12th power. If the factors are on different sides, the product will be the excess on the side of the greater.—So, the product of 1 divided by Mal-i-Mal into Mal-i-Cab, is Shai; and the product of 1 divided by Cab-i-Cab-Cab into Cab-i-Mal-i-Mal, is 1 divided by Mal. And if the factors are at the same distance (from 1), the product is 1. The author adds, "The particulars of the methods of division, with extraction of roots, and other rules, I have given in my greater book."

"The rules of algebra which have been discovered by learned men are six, and they relate to number and Shai and Mal." That is, to n and x and x^2, including what we call simple and quadratic equations. "The following table will show the products and quotients of these, which are here given for the sake of brevity."

	Multiplier.					
	$\frac{1}{x^2}$	$\frac{1}{x}$	1	x	x^2	
Dividend.						**Multiplicand.**
x^2	1	x	x^2	x^3	x^4	x^2
x	$\frac{1}{x}$	1	x	x^2	x^3	x
1	$\frac{1}{x^2}$	$\frac{1}{x}$	1	x	x^2	1
$\frac{1}{x}$	$\frac{1}{x^3}$	$\frac{1}{x^2}$	$\frac{1}{x}$	1	x	$\frac{1}{x}$
$\frac{1}{x^2}$	$\frac{1}{x^4}$	$\frac{1}{x^3}$	$\frac{1}{x^2}$	$\frac{1}{x}$	1	$\frac{1}{x^2}$
	x^2	x	1	$\frac{1}{x}$	$\frac{1}{x^2}$	
	Divisor.					

The use of the table is this: multiply the coefficients of the two quantities together, for the coefficient of the product, which is of the denomination contained in the square where the lines from the two factors meet. If on either side there be a subtractive (negative) quantity, call the minuend plus or affirmative, and the subtrahend minus or negative. The product of plus into plus, and of minus into minus, are both plus, and the product of different kinds is minus. Multiply the quantities together, and subtract the negative from the affirmative. For example, the product of 10 and 1 Shai $(10 + x)$ into 10 wanting 1 Shai $(10 - x)$, is 100 wanting 1 Mal $(100 - x^2)$. The product of 5 wanting 1 Shai $(5 - x)$, by 7 wanting Shai $(7 - x)$, is 35 and one Mal wanting 12 Shai $(35 + x^2 - 12x)$. Also, the product of 4 Mal and 6 wanting 2 Shai $(4x^2 + 6 - 2x)$, into 3 Shai wanting 5 $(3x - 5)$, is 12 Cab and 28 Shai wanting 26 Mal and 30 $(12x^3 + 28x - 26x^2 - 30)$. In division, find a number which multiplied by the divisor will produce the dividend; divide the coefficient of the dividend by that of the divisor, the quotient is the coefficient of the quantity which is opposite to the dividend and divisor.

CHAPTER II. *On the Six Rules of Algebra.*

To find unknown quantities by algebra depends on acuteness and sagacity, with an attentive consideration of the terms of the question, and a successful application of the invention to such things as may serve to bring out the quantity required. Call the unknown quantity Shai (the Thing), and proceed with it according to the terms of the question, till the operation ends with an equation. Let that side where there are negative quantities be made perfect, adding the negative quantity to the other side, which is called restoration (*Jebr**). Let those things which are of the same kind, and equal on both sides, be thrown away, which is called opposition (*Makabulah*).

Equality is either of one species to another, which is of three kinds, called simple (*Mufridat*), or of one species to two species, which is of three kinds, called compound (*Muktarinat*).

CASE 1st *Mufridat.* When number is equal to *things* ($ax = n$); divide the number by the coefficient of the thing, and the unknown quantity will be found. *Example.* A person admitted that he owed to Zaid 1000 and one half of what he owed Amer; also that he owed Amer 1000 all but half of what he owed to Zaid. Call Zaid's debt Shai (x). Then Amer's debt is 1000 wanting half of Shai ($1000-\frac{1}{2}x$); and Zaid's is 1500 wanting a fourth of Shai ($1500-\frac{1}{4}x$). This is equal to Shai ($1500 - \frac{1}{4}x = x$). After *Jebr*, 1500 is equal to one Shai and a quarter of Shai ($1500 = 1\frac{1}{4}x$). So for Zaid is 1200, and for Amer 400.

CASE 2nd. *Multiples of Shai equal to multiples of Mal* ($ax = bx^2$). Divide the coefficient of the thing by that of Mal; the quotient is the unknown quantity ($\frac{a}{b} = x$). *Example.* Some sons plundered their father's inheritance, which consisted of Dinars. One took 1, another 2, the

* Hence doubtless, with the article *Al*, comes the name *Algebra.*

third 3, and so on increasing by 1. The ruling power took back what they had plundered, and divided it among them in equal shares, by which each received 7. How many sons were there, and how many Dinars? Suppose the number of sons Shai (x), and take the sum of the extremes, viz, 1 and Shai ($1 + x$). Multiply them by half of Shai ($\frac{1}{2}x$); this is the number of dinars ($\frac{1}{2}x + \frac{1}{2}x^2$); for the sum of any series of numbers in arithmetical progression, is equal to the product of the sum of the two extremes, into half the number of terms. Divide the number of the Dinars by Shai (x), which is the number of the sons, the quotient, according to the terms of the question, will be 7 ($(\frac{1}{2}x^2 + \frac{1}{2}x) \div x = 7$). Multiply 7 by Shai ($x$), the divisor, 7 Shai is the product, which is equal to $\frac{1}{2}$ Mal and $\frac{1}{2}$ Shai ($7x = \frac{1}{2}x^2 + \frac{1}{2}x$). After *Jebr* and *Mukabalah*, 1 Mal is equal to 13 Shai ($x^2 = 13x$); Shai then is 13 ($x = 13$); and this is the number of sons. Multiply this by 7, and the number of Dinars will be found 91.

It is added, that questions of this sort may be solved by position. "Thus, suppose the number of sons to be 5, the first error is 4 in defect; then suppose it to be 9, the second error is 2 in defect. The first *Mafudh* is 10, and the second is 36; their difference is 26, and the difference of the errors is 2."—"Another method, which is short, is this: Double the quotient (7) is 14; subtract 1, and the result (13) is the number of sons."

"CASE 3d. *Number equal to Mal* ($n = ax^2$). Divide the number by the coefficient of the Mal; the root of the quotient ($\sqrt{\frac{n}{a}} = x$) is the unknown quantity. *Example.* A person admitted that he owed Zaid the greater of two sums of money, the sum of which was 20, and the product 96. Suppose one of them to be 10 and Shai ($10 + x$), and the other 10 wanting Shai ($10 - x$). The product, which is 100 all but Mal ($100 - x^2$) is equal to 96; and after *Jebr* and *Mukabalah*, 1 Mal is equal to 4 ($x^2 = 4$), and Shai equal to

$2(x = 2)$. One of the sums then is 8, and the other 12, which is the debt of Zaid.

Here, by substituting for the half-sum and half-dif. the equation comes out a simple one.

First Case of *Maktarinat*. Number equal to Mal and Shai ($ax^2 + bx = n$). Complete the Mal to unit if it is deficient, or reduce it to the same if it exceeds, and reduce the number and Shai in the same ratio, by dividing all by the coefficient of the Mal. Then square one half the coefficient of the Shai, and add this square to the number. From the root of the sum subtract half the coefficient of the Shai, and the unknown will remain. An example follows as usual: and the method is evidently the same as ours at present.

Case 2*nd*. Shai equal to number and Mal ($bx = x^2 + n$). After completing or rejecting, subtract the number from the square of half the coefficient of Shai; then add the root of the remainder to half the coefficient of the Shai, or subtract the former from the latter, and the result is the unknown quantity. This also is the same as the present method, and both the two roots are noticed in this case, by taking the root of the known quantity either positive or negative. An example is added as usual.

Case 3*d*. Mal equal to number and Shai ($x^2 = n + bx$). After completion or rejection, add the square of half the coefficient of the Shai to the number, and add the root of the sum to half the coefficient of the Shai; this is the unknown quantity. *Example*. What number is that which being subtracted from its square, and the remainder added to its square, is 10? Subtract Shai from Mal and go on with the operation, then 2 Mal all but Shai is equal to 10 ($2x^2 - x = 10$); and after *Jebr* and *Radd*, Mal is equal to 5 and half of Shai ($x^2 = 5 + \frac{1}{2}x$). The square of half the coefficient of Shai and 5, is 5 and half an eighth ($5\frac{1}{16}$), and its root is $2\frac{1}{4}$; to this add $\frac{1}{4}$, the result $2\frac{1}{2}$ is the number sought.

Book 9*th*, contains 12 rules respecting the properties of

numbers. As, 1st. To find the sum of the products multiplied into itself and into all numbers below it. Add 1 to the number, and multiply the sum by the square of the number; half the product is the number sought.

2nd. To add the odd numbers in their regular order. Add 1 to the last number, and take the square of half the sum.

3d. To add the even numbers from 2 upwards. Multiply half the last even number by a number greater by 1 than that half.

4th. To add the squares of the numbers in order. Add 1 to twice the last number, and multiply a 3d of the sum by the sum of the numbers.

5th. To find the sum of the cubes in succession. Take the square of the sum of the numbers.

6th. To find the product of the roots of two numbers. Multiply one by the other, and the root of the product is the answer.

7th. To divide the root of one number by that of another. Divide one by the other, the root of the quotient is the answer.

8th. To find a perfect number; that is, a number which is equal to the sum of its aliquot parts, (Eucl. book 7, def. 22). The rule is that delivered by Euclid, book 9, prop. 36.

9th. To find a square in a given ratio to its root. Divide the first number of the ratio by the second; the square of the quotient is the square required.

10th. If any number be multiplied and divided by another, the product multiplied by the quotient, is the square of the first number.

11th. The difference of two squares, is equal to the product of the sum and difference of the roots.

12th. If two numbers be divided by each other, and the quotients multiplied together, the result is always 1.

Book 10th, contains 9 practical examples, all of which are capable of solution by simple equations, or by position, or by retracing the steps of the operation, and some of them

by simple proportion; so that it is needless to specify them.

The conclusion, which marks the limits of algebraical knowledge in the age of the writer, is here given entire, in the author's words.

"CONCLUSION. There are many questions in this science which learned men have to this time in vain attempted to solve; and they have stated some of these questions in their writings, to prove that this science contains difficulties, to silence those who pretend they find nothing in it above their ability, to warn arithmeticians against undertaking to answer every question that may be proposed, and to excite men of genius to attempt their solution. Of these I have selected seven.—1st. To divide 10 into two parts, such, that when each part is added to its square root, and the sums are multiplied together, the product is equal to a supposed number.—2d. What square number is that, which being increased or diminished by 10, the sum and remainder are both square numbers?—3d. A person said he owed to Zaid 10 all but the square root of what he owed to Amer, and that he owed Amer 5 all but the square root of what he owed Zaid.—4th. To divide a cube number into two cube numbers.—5th. To divide 10 into two parts such, that if each is divided by the other, and the two quotients are added together, the sum is equal to one of the parts.—6th. There are three square numbers in continued geometric proportion, such, that the sum of the three is a square number.—7th. There is a square, such, that when it is increased and diminished by its root and 2, the sum and the difference are squares.—Know, reader, that in this treatise I have collected in a small space the most beautiful and best rules of this science, more than were ever collected before in one book. Do not underrate the value of this bride; hide her from the view of those who are unworthy of her, and let her go to the house of him only who aspires to wed her."

From the preceding account of this Arabian treatise on

Algebra, the Khulasat-ul-Hisab, of Baha-ul-din, which is esteemed the best in that language, we pretty clearly perceive what was the state of the science in that nation; that it was much the same as the first treatises among the Italians, derived directly from the former; but that it was much inferior to the same science among the Indians. It does not appear that the Arabians used algebraic notion or abbreviating symbols; that they had any knowledge of the Diophantine algebra, or of any but the easiest and elementary parts of the science. We have seen that Baha-ul-din ascribes the invention of the numeral figures, in the decimal scale, to the Indians; as is done indeed by all the Arabic and Persian books of arithmetic. The following is an extract from a Persian treatise of arithmetic in Mr. Strachey's possession.

" The Indian sages wishing to express numbers conveniently, invented these nine figures, ١ ٢ ٣ ۴ ۵ ۶ ٧ ٨ ٩. The first figure on the right hand they made stand for units, the second for tens, the third for hundreds, the fourth for thousands. Thus, after the third rank, the next following is units of thousands, the second tens of thousands, the third hundreds of thousands, and so on. Every figure therefore in the first rank is the number of units it expresses; every figure in the second, the number of tens which the figure expresses; in the third, the number of hundreds; and so on. When in any rank a figure is wanting, write a cipher like a small circle ( ) to preserve the rank. Thus, ten is written 10, a hundred 100, five thousand and twenty-five 5025."

In short, of the Indian algebra, in its full extent, the Arabians seem to have been ignorant; but most likely they had their algebra from the same source as their arithmetic. The Arabian and Persian treatises on algebra, like the first and old European ones, begin with arithmetic, called in those treatises the arithmetic of the Indians, and have a second part on algebra; but no notice is taken of the origin

of the latter. Most likely their algebra, being numeral, was considered by the authors as a part of arithmetic.

Though part only of the Khulasat-ul-Hisab, Mr. Strachey says, is concerning algebra, the rest, relating to arithmetic and mensuration, must be thought not wholly unconnected with the subject. It is to be hoped that ere long we shall have either translations from the Sanscrit of the Bija Ganita and Lilawati, or perfect accounts from the originals; and that other Hindu books of algebra will be found, and made known to the world. But in the mean time the Persian translations will be found well deserving of attention; observing carefully to distinguish between what is interpolated, and what is original.

From the preceding account of this Arabic treatise, also, is clearly seen the origin of the name of Algebra, being an Arabic compound, viz. of the article *al*, and *jebr*, which denotes one of the modes of reducing the equations, viz. by transposing or adding the negative terms, to make them all affirmative.

From the circumstance of receiving the notices of the communications on the foregoing subjects at several different times, while the account was composing, and even while printing, it is hoped that some repetitions, and some irregularities in the composition, may be candidly excused.

OF ALGEBRA IN ITALY AND OTHER PARTS OF EUROPE.

We have seen that algebra had probably its rise in Hindustan, as well as our present numeration and arithmetic, all the rules of it having been found in the ancient MSS. of that country, much the same, in matter and form, as they appear in the first Italian authors; and among these, the two rules of false position, which are nearly allied to algebra, and the extension of which probably led to, and ultimately became the art of algebra itself. Hence it appears that these arts passed successively into Persia, Arabia, Afri-

ca, and Europe. In this last quarter it seems doubtful, whether their introduction was first into Spain or into Italy: the probability would appear to be on the side of the former, as it would most likely be introduced by the Moors on their settlement in the peninsula in the 8th, 9th, and 10th centuries; whence it might be communicated to Gaul and Germany and England, &c. This route is rendered the more probable by the circumstance, that the early state of the algebraic art in these countries was very different, and even more perfect, in several respects, than that in Italy, in the contemporary stages. And yet, on the other hand, against this probability of the first introduction of the art into Spain, by means of the Moorish conquest, it may reasonably be objected, that it does not appear that the Arabians themselves had cultivated it at a time so early as the date of that conquest; and besides that we have not heard of any early works on Algebra having ever been found in the Spanish peninsula. It appears, however, at any rate, that Italy received the art by a different route, though probably at a later date, viz, immediately from the eastern Arabians themselves, without the knowledge of any other source or communication: and hence it was natural for the Italians to ascribe the invention of the art to that people.

It has usually been thought, that the first introduction of algebra into Italy, was about the 14th century. But we have lately, viz, in 1797, been favoured with a new and very diffuse history of algebra, by Sig. Cossali, in two large volumes, 4to, by which it appears, that the art was first imported into Italy, from the east, by Leonard Bonacci, of Pisa, who composed his arithmetic in the year 1202, and again in 1228; adding the algebra at the end, as a part of it.

In this work of Cossali's, accounts are given of several other old authors, as well Italian as Arabian and Persian, &c, among which some were much earlier than Leonard: as, a Mohamed ben Musa or Moses, called also Mohamed of Buziani, a place in Corasan, near the south-east point of

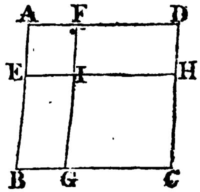
the Caspian Sea. This Mohamed, it seems, in the year 959, travelled eastward to the confines of India, to learn the mathematical sciences, and afterwards to teach them; and who, according to Abulfaragio, in the year 969, wrote a Commentary on Diophantus; he wrote also Demonstrations of the Propositions in the same; and another work on the Universal Logistic Art, in three books.

It seems, however, that it was Leonard Bonacci, of Pisa, who first introduced the art in Italy, as before-mentioned. Leonard's work, it appears, was a very orderly and regular treatise on arithmetic and algebra, as far as it was then known; teaching and demonstrating all the rules, and illustrating them with many examples; being also much occupied with questions about square and cube numbers, like those of Diophantus, or rather like what we have seen and described among the Indians and Arabians.—In algebra, Leonard distinguished three kinds of numbers, viz, the absolute known number in any question; then the unknown number, which he calls *radice*, the root; and its square, which he calls *census* in Latin, or *censo* in Italian; for his algebra extended only to the solution of equations of the 1st and 2d degree, the same as that of the Indians and Arabians. The language in which the work was written, was in barbarous Latin, or something between Latin and Italian, when the language of the country was changing from the one to the other.

In treating his subject, Leonard had no such notation as is used by modern authors; on the contrary, he expressed every thing, both the quantities and the several operations, by their names, or words at full length. Those equations he treated of in six different forms, which are as follow, when expressed in the modern notation. 1st, $x^2 = ax$; 2d, $x^2 = n$; 3d, $ax = n$; 4th, $x^2 + ax = n$; 5th, $x^2 = ax + n$; 6th, $x^2 + n = ax$; where x denotes the *radice*, or quantity sought, x^2 its *censo* or square, a the number multiplying x, and n the absolute known number; and where it is remarkable that he places the terms, more or less,

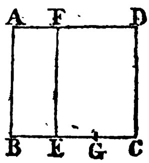
on the one side or the other, so as they may be all affirmative: all which cases and forms are exactly the same as before given by their masters the Arabians, &c.

Now omitting the simple forms, as including but little remarkable, we shall only consider the compound ones, or the solution of quadratic equations, by the method of completing the square, which Leonard founds on geometrical demonstration, as follows:—In the first case, if $x^2 + ax = n$, then the rule is $x = \sqrt{(\frac{1}{4}a^2 + n)} - \frac{1}{2}a$; which he thus demonstrates.



Demonstr. If upon a right line BC, greater than $\frac{1}{2}a$, the square ABCD be constructed, on the sides of which there are taken the parts CG, CH, DF, BE, each equal to $\frac{1}{2}a$, and drawing the right lines EH, FG, intersecting in I, they form the square IGCH $= \frac{1}{4}a^2$. Supposing EI = AF = EA = IF = BG = DH, to denote the required quantity x; then will the square AI or of EI $= x^2$, the rectangle BI $= \frac{1}{2}ax =$ also to the rectangle ID. Therefore the whole square ABCD $= x^2 + ax + \frac{1}{4}a^2$: but $x^2 + ax = n$; therefore is the same square ABCD $= n + \frac{1}{4}a^2$: consequently the side BC $= \sqrt{(n + \frac{1}{4}a^2)}$; and the quantity sought $x =$ BC − GC = BC $- \frac{1}{2}a = \sqrt{(n + \frac{1}{4}a^2)} - \frac{1}{2}a$.

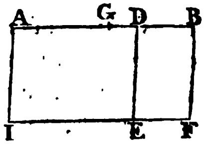
2ndly. If $x^2 = ax + n$; then will $x = \sqrt{(\frac{1}{4}a^2 + n)} + \frac{1}{2}a$.



Demonstr. Let the right line BC $= x$, and the square upon it ABCD $= x^2$. If CE be taken $= a$; and the perp. EF $= x$. Then the rectangle ED $= ax$. Therefore the remaining rectangle AE $= n$. But AE = AB × BE = BC × BE; therefore n = BC × BE = BE^2 + BE × EC. If EC be bisected in G, then will $n +$ EG^2 = BE^2 + BE × EC + EG^2. Hence $\sqrt{(n + \text{EG}^2)}$ = BE + EG; and GC + $\sqrt{(n + \text{EG}^2)}$ = GC + BG. So that, it being GC = EG = $\frac{1}{2}a$, and BC $= x$, it will be $\frac{1}{2}a + \sqrt{(n + \frac{1}{4}a^2)} = x$.

3dly. Let $x^2 + n = ax$.—Here he says, if $\frac{1}{4}a^2 < n$, the

equation is impossible.—If $\frac{1}{4}a^2 = n$, then is $x = \frac{1}{2}a$.—If $\frac{1}{4}a^2 > n$, then is $x = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - n)}$, or $= \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - n)}$.



Demonstr. Draw AB representing a, bisected in G, and unequally divided in D; and upon one of the unequal parts, supposed $= x$, if there be made a square $= x^2$. Now, if, in the first place, it be formed on the less part DB, supposed $= x$; let FE be produced to I, till FI be = AB, and let AI be drawn. Then the whole rectangle AF = AB × BF = ax; hence, taking away the square BE $= x^2$, there remains the rectangle AE = AD × DE = AD × DB $= n$; and adding to both the square GD^2, gives AD × DB + $\text{GD}^2 = n + \text{GD}^2$. But, by Eucl. II. 5, it is AD × DB + $\text{GD}^2 = \text{BG}^2$; therefore $\text{BG}^2 = n + \text{GD}^2$; hence $\text{BG}^2 - n = \text{GD}^2$, and $\sqrt{(\text{BG}^2 - n)} = \text{GD}$, then $\text{BG} - \sqrt{(\text{BG}^2 - n)} = \text{BG} - \text{GD} = \text{DB}$, or $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - n)} = x$. —If the square x^2 be constituted on the greater part AD, considered as $= x$; then by a like process it will be $\sqrt{(\text{AG}^2 - n)} = \text{GD}$; hence $\text{AG} + \sqrt{(\text{AG}^2 - n)} = \text{AG} + \text{GD} = \text{AD}$, that is $\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - n)} = x$.

Thus we see that Leonard derived the rules for quadratic equations from geometrical considerations, and even the double values in the possible case of $x^2 + n = ax$; as it is probable his predecessors had done, and as we find his successors Lucas de Burgo and others did also. And this method shows the reason why these writers never admitted those values of x, in which n is greater than $\frac{1}{4}a^2$. We further find, that neither Leonard, nor the Arabians, nor Indians, made any use of the form $x^2 + ax + n = 0$, nor of the negative roots of the two forms $x^2 + ax = n$, and $x^2 = ax + n$: it being Harriot who first departed from the usual custom (of equalizing the positive terms to the positive in an equation), by placing all the terms on one side, with their signs + or −, and made = 0 on the other side, thus $x^2 \pm ax \pm n = 0$; and it was Cardan who had the honour of showing that, of the two forms $x^2 + ax = n$, and

$x^2 = ax + n$, the negative root of the one is the positive root of the other.

It appears that Leonard Pisanis was well skilled in the various ways of reducing equations to their final simple state, by all the usual methods; as addition, subtraction, multiplication, division, powers and roots, to free them from radicals, &c. He was also well acquainted with the modes of substitution, so as to bring out the equation in the lowest degree; such as, in cases of two unknown quantities, instead of finding either of them separately, he would first search out their sum or difference. So, for instance, in this problem, " To divide the number 10 into two such parts, that from the greater part taking its own square-root, and to the less part adding its square root, the two results should be equal." Now, in this case, if in the common way x be made to denote one of the parts, as the greater suppose, then $10-x$ will denote the less part, and the equation will be $x-\sqrt{x}=10-x+\sqrt{(10-x)}$, the reduction of which leads to a complete equation of the 4th degree. But, instead of this method, Leonard employs the way of the half-sum and half-difference, as used by Diophantus and the Arabians and Hindus, which has been accounted by some persons an artifice of the modern algebraists. Thus, if x be put to denote the half-difference of the two parts; then, 5 being their half-sum, $5+x$ will be the greater, and $5-x$ the less, and the equation will be $5+x-\sqrt{(5+x)}= 5-x+\sqrt{(5-x)}$, which reduces to a final quadratic, as is done by this author.

From Pisa, as from a centre, it seems the art gradually spread through Tuscany and all Italy; in consequence of which, many other authors in that country had respectable names before the period of the art of printing. So, we read that Raffaello Canacci, a Florentine mathematician, was author of a " Ragionamento di Algebra," who praises another that preceded him, named Guglielmo di Lunis; of whom Canacci writes, at the beginning of his Ragionamento, " La regola dell' algibra, la quale regola Ghughelmo di

Lunis la translato d'Arabico a nostra lingua;" whence some have thought, though without probability, that the honour of having made Italy acquainted with algebra, was due to Guglielmo di Lunis, rather than to Leonardo Bonacci di Pisa. Many other early authors are mentioned. Bombelli, in the preface to his book, writes that a work of Mohamed ben Musa had been shown, but it was of little value. And it is said that Lucas de Burgo, the first author in print, was instructed in this science at Venice, by Domanico Bragadini; successor in the public chair to the learned Paolo della Pergola, his preceptor, who died in 1366.

Proceed we now to the consideration of the books of Lucas de Burgo, and other authors, whose works we are possessed of in print.

LUCAS PACIOLUS, OR DE BURGO.

It was chiefly among the Italians that this art was first cultivated in Europe. And the first author whose works were in print, was Lucas Paciolus, or Lucas de Burgo, a cordelier, or minorite friar. He wrote several treatises of arithmetic, algebra, and geometry, which were printed in the years 1470, 1476, 1481, 1487, and in 1494 his principal work, intitled Summa de Arithmetica, Geometria, Proportioni, et Proportionalita, which is a very masterly and complete treatise on those sciences, as they then stood. In this work he mentions various former writers, as Euclid, St. Augustine, Sacrobosco or Halifax, Boetius, Prodocimo, Giordano, Biagio da Parma, and Leonardus Pisanus, from whom he learned those sciences.

The order of the work is, 1st arithmetic, 2d algebra, and 3d geometry. Of the arithmetic, the contents, and the order of them, are nearly as follow. First, of numbers figurate, odd and even, perfect, prime and composite, and many others. Then of common arithmetic in seven parts, namely, numeration or notation, addition, subtraction, multiplication, division, progression, and extraction of roots. Before him, he says, duplation and mediation, or doubling

and halving, were accounted two rules in arithmetic; but that he omits them, as being included in multiplication and division. He ascribes the present notation and method of arithmetic to the Arabs; and says that according to some the word Abaco is a corruption of Modo Arabico, but that according to others it was from a Greek word. This, however, must be a mistake; for though the Italians had those arts from the Arābs, these latter had them probably from the Indians. All those primary operations he both performs and demonstrates in various ways, many of which are not in use at present; proving them not only by what is called casting out the nines, but also by casting out the sevens, and otherwise. In the extraction of roots he uses the initial ℞ for a root; and when the roots can be extracted, he calls them discrete or rational; otherwise surd, or indiscrete, or irrational. The square root is extracted much the same way as at present, namely, dividing always the last remainder by double the root found; and so he continues the surd roots continually nearer and nearer in vulgar fractions. Thus, for the root of 6, he first finds the nearest whole number 2, and the remainder 2 also; then $\frac{2}{4}$ or $\frac{1}{2}$ is the first correction, and $2\frac{1}{2}$ the second root: its square is $6\frac{1}{4}$, therefore $\frac{1}{4}$ divided by 5, or $\frac{1}{20}$, is the next correction, and $2\frac{1}{2}$ minus $\frac{1}{20}$, or $2\frac{9}{20}$ is the 3d root: its square is $6\frac{1}{400}$, therefore $\frac{1}{400}$ divided by $4\frac{9}{10}$, or $\frac{1}{1960}$, is the 3d correction, which gives $2\frac{881}{1960}$ for the 4th root, whose square exceeds 6 by only $\frac{1}{3841600}$: and so on continually: and this process he calls approximation. He observes that fractions, which he sets down the same way as we do at present, are extracted, by taking the root of the denominator, and of the denominated, for so he calls the numerator: and when mixed numbers occur, he directs to reduce the whole to a fraction, and then extract the roots of its two terms as above: as if it be $12\frac{1}{4}$; this he reduces to $\frac{49}{4}$, and then the roots give $\frac{7}{2}$ or $3\frac{1}{2}$: in like manner he finds that $4\frac{1}{2}$ is the root of $20\frac{1}{4}$; $5\frac{1}{2}$ the root of $30\frac{1}{4}$; "and so on (he adds) in infinitum;" which shows that he knew how to form the series of squares by addition.

He then extracts the cube root, by a rule much the same as that which is used at present; from which it appears that he was well acquainted with the co-efficients of the binomial cubed, named, 1, 3, 3, 1; and he directs how the operation may be continued "in infinitum" in fractions, like as in the square root. After this, he describes geometrical methods for extracting the square and cube roots instrumentally: he then treats professedly of vulgar fractions, their reductions, addition, subtraction, and other operations, much the same as at present: then of the rule-of-three, gain-and-loss, and other rules used by merchants.

Paciolus next enters on the algebraical part of this work, which he calls "L'Arte Magiore; ditta dal vulgo la Regola de la Cosa, over Alghebra e Almucabala:" which last name he explains by *restauratio & oppositio*, and assigns as a reason for the first name, that it treats of things above the common affairs in business, which make the *arte minore*. Here he mistakenly ascribes the invention of algebra to the Arabians; and he says that the Arabic algebra means in Italian position, or rather opposition. He denominates the series of powers, with their marks or abbreviations, as *n°.* or *numero*, the absolute or known number; *co.* or *cosa*, the thing or 1st power of the unknown quantity; *ce.* or *censo*, the product or square; *cu.* or *cubo*, the cube, or 3d power; *ce. ce.* or *censo de censo*, the square-squared, or 4th power; *p°. r°.* or *primo relato*, or 5th power; *ce. cu.* or *censo de cubo*, the square of the cube, or 6th power; and so on, compounding the names or indices according to the multiplication of the numbers 2, 3, &c, and not according to their sum or addition, as used by Diophantus, with the Arabians and Indians. He describes also the other characters made use of in this part, which are for the most part no more than the initials or other abbreviations of the words themselves, after the manner of the Indians; as for ℞ *radici*, the root; ℞. ℞. *radici de radici*, the root of the root; ℞ u. *radici universale*, or *radici legata*, or *radici unita*; ℞ *cu. radici cuba*; and q̄3 *quantita*, quantity; *p* for *piu* or plus, and *m* for *meno*

or minus; and he remarks that the necessity and use of these two last characters are for connecting, by addition or subtraction, different powers together; as 3 *co.* p. 4 *ce.* m. 5 *cu.* p. 2 *ce. ce.* m. 6 *n*ⁱ. that is, 3 *cosa*, piu 4 *censa*, meno 5 *cubo*, piu 2 *censa-censa*, meno 6 *numeri*, or, as we now write the same thing, $3x + 4x^2 - 5x^3 + 2x^4 - 6$.

He first treats very fully of proportions and proportionalities, both arithmetical and geometrical, accompanied with a large collection of questions concerning numbers in continued proportion, resolved by a kind of algebra. He then treats of *el Cataym*, which he says, according to some, is an Arabic or Phenician word, and signifies the Double Rule of False Position: but he here treats of both single and double position, as we do at present, dividing the *el Cataym* into single and double, as the Arabians and Indians did. He gives also a geometrical demonstration of both the cases of the errors in the double rule, namely, when the errors are both plus or both minus, and when the one error is plus and the other minus; and adds a large collection of questions, as usual. He then goes through the common operations of algebra, with all the variety of signs, as to plus and minus; proving that, in multiplication and division, like signs give plus, and unlike signs give minus. He next treats of different roots *in infinitum*, and the extraction of roots; giving also a copious treatise on radicals or surds, as to their addition, subtraction, multiplication, and division, and that both in square roots and cube roots, and in the two together, much the same as at present. He makes here a digression concerning the 15 lines in the 10th book of Euclid, treating them as surd numbers, and teaching the extraction of the roots of the same, or of compound surds or binomials, such as of 28 *p* ℞ 448, or of ℞ 18 *p* ℞ 10; and gives this rule, among several others, namely: Divide the first term of the binomial into two such parts, that their product may be $\frac{1}{4}$ of the number in the second term; then the roots of those two parts, connected by their proper sign *p* or *m*, is the root of

the binomial; as in this 23 *p* ℞ 448, the two parts of 23 are 7 and 16, whose product, 112, is $\frac{1}{4}$ of 448, therefore their roots give 4 *p* ℞ 7 for the root ℞ u. 23 *p* ℞ 448.

He next treats of equations both simple and quadratic, or simple and compound, as he calls it; and this latter he performs by completing the square, and then extracting the root, just as was usual. He also resolves equations of the simple 4th power, and of the 4th combined with the 2d power, which he treats the same way as quadratics; expressing his rules in a kind of bad verse, and giving geometrical demonstrations of all the cases, the same as those of Leonard of Pisa, before described. He uses both the roots or values of the unknown quantity, in that case of the quadratics which has two positive roots; but he takes no notice of the negative roots in the other two cases. As to any other compound equations, such as the cube and any other power, or the 4th and 1st, or 4th and 3d, &c, he gives them up as impossible, or at least he says that no general rule has yet been found for them, any more, he adds, than for the quadrature of the circle.—The remainder of this part is employed on rules in trade and merchandize, such as Fellowship, Barter, Exchange, Interest, Composition or Alligation, with various other cases in trade. And in the third part of the work, he treats of Geometry, both theoretical and practical.

From this account of Lucas de Burgo's book, we may perceive what was the state of algebra about the year 1500, in Europe; and probably it was much the same in Africa and Arabia, from whence the Europeans had it. It appears that their knowledge extended no further than quadratic equations, of which they used only the positive roots; that they used only one unknown quantity; that they had no marks or signs for either quantities or operations, excepting only some few abbreviations of the words or names themselves; and that the art was only employed in resolving certain numeral problems. So that either the orientals had not carried algebra beyond quadratic equations, or else the

Europeans had not learned the whole of the art, as it was then known to the former. And indeed it is not impossible but this might be the case: for whether the art was brought to us by an European, who, travelling into the East, there learned it; or whether it was brought to us by those people; in either case we might receive the art only in an imperfect state, and perhaps far short of the degree of perfection to which it had been carried by their best authors. And this suspicion is rendered rather probable by the circumstance of an Arabic manuscript, said to be on cubic equations, deposited in the library of the university of Leyden by the celebrated Warner, bearing a title, which in Latin signifies *Omar Ben Ibrahim al'Ghajamæi Algebra cubicarum æquationum, sive de problematum solidorum resolutione.* At any rate, however, we have found that the Hindus, if not the Arabians, treated of problems including several unknown quantities, with applications of algebra to geometrical and to indeterminate problems.

FERREUS AND CARDAN.

After the publication of the books of Lucas de Burgo, the science of algebra became more generally known, and improved, especially by many persons in Italy; and about this time, or soon after, namely, about the year 1505, the first rule was there found out by Scipio Ferreus, for resolving one case of a compound cubic equation. But this science, as well as other branches of mathematics, was most of all cultivated and improved there by Hieronymus Cardan of Bononia, a very learned man, whose arithmetical writings were the next that appeared in print, namely, in the year 1539, in nine books, in the Latin language, at Milan, where he practised physic, and read public lectures on mathematics; and in the year 1545 came out a 10th book, containing the whole doctrine of cubic equations, which had been in part revealed to him by Tartalea, about the time of the publication of his first nine books. And as it is

only this 10th book which contains the new discoveries in algebra, I shall here confine myself to it alone, as it will also afford sufficient occasion to speak of his manner of treating algebra in general. This book is divided into 40 chapters, in which the whole science of cubic equations is most amply and ably treated. Chap. 1, treats of the nature, number, and properties, of the roots of equations, and particularly of single equations that have double roots. He begins with a few remarks on the invention and name of the art: calls it *Ars Magna*, or *Cosa*, or *Rules of Algebra*, after Lucas de Burgo and others: says it was invented by Mahomet, the son of one Moses, an Arabian, as is testified by Leonardus Pisanus; and that he left four rules or cases, which only included quadratic equations: that afterwards three derivatives were added by an unknown author, though some think by Lucas Paciolus; and after that again three other derivatives, for the cube and 6th power, by another unknown author; all which were resolved like quadratics: that then Scipio Ferreus, professor of mathematics at Bononia, about 1505, found out the rule for the case *cubum & rerum numero æqualium*, or, as we now write it, $x^3 + bx = c$, which he speaks of as a thing admirable: that the same thing was next afterwards found out, in 1535, by Tartalea, who revealed it to him, Cardan, after the most earnest intreaties: that, finally, by himself and his quondam pupil Lewis Ferrari, the cases were greatly augmented and extended, namely, by all that is not here expressly ascribed to others; and that all the demonstrations of the rules are his own, except only three adopted from Mahomet for the quadratics, and two of Ferrari for cubics.

Cardan then delivers some remarks, showing that all square numbers have two roots, the one positive, and the other negative, or, as he calls them, *vera & ficta*, true and fictitious or false; so the *æstimatio rei*, or root, of 9, is either 3 or -3; of 16 it is 4 or -4; the 4th root of 81 is 3 or -3; and so on for all even *denominations* or powers. And the same is remarked on compound cases of even

powers that are added together; as if $x^4 + 3x^2 = 28$, then the æstimatio x is $+2$ or -2; but that the form $x^4 + 12 = 7x^2$ has four answers or roots, in real numbers, two plus and two minus, viz, 2 or -2, and $\sqrt{3}$ or $-\sqrt{3}$; while the case $x^4 + 12 = 6x^2$ has no real roots; and the case $x^4 = 2x^2 + 8$ has two, namely, 2 and -2: and in like manner for other even powers. So that he includes both the positive and negative roots; but rejects what we now call imaginary ones. I here express the cases in our modern notation, for brevity sake, as he commonly expresses the terms by words at full length, calling the absolute or known term the *numero*, the 1st power the *res*, the 2d the *quadratum*, the 3d the *cubum*, and so on, using no mark for the unknown quantity, and only the initials *p* and *m* for plus and minus, and ℞ for radix or root. The *res* he sometimes calls *positio* (as derived from the like quantity in the rule of position), and *quantitas ignota*; and in stating or setting down his equations, he, as well as Lucas de Burgo before him, sets down the terms on that side where they will be plus, and not minus.

On the other hand, he remarks that the odd denominations, or powers, have only one æstimatio, or root, and that true or positive, but none fictitious or negative, and for this reason, that no negative number raised to an odd power, will give a positive number; so of $2x = 16$, the root is 8 only; and if $2x^3 = 16$, the root is 2 only: and if there be ever so many odd denominations, added together, equal to a number, there will be only one æstimatio or root; as if $x^3 + 6x = 20$, the only root is 2. But that when the signs of some of the terms are different, as to plus and minus, they may have more roots; and he shows certain relations of the coefficients, when they have two or more roots: so the equation $x^3 + 16 = 12x$ has two æstimatios, the one true or 2, and the other fictitious or -4, which he observes is the same as the true æstimatio of the case $x^3 = 12x + 16$, having only the sign of the absolute number changed from the former, the 3d root 2 being the same as

the first, which therefore he does not count. He next shows what are the relations of the coefficients when a cubic equation has three roots, of which two are true, and the 3d fictitious, which is always equal to the sum of the other two, and also equal to the true root of the same equation with the sign of the absolute number changed: thus, in the equation $x^3 + 9 = 12x$, the two true roots are 3 and $\sqrt{5\frac{1}{4}} - 1\frac{1}{2}$, and the fictitious one is $-\sqrt{5\frac{1}{4}} - 1\frac{1}{2}$, which last is the same as the true root of $x^3 = 12x + 9$, viz, $\sqrt{5\frac{1}{4}} + 1\frac{1}{2}$; and he here infers generally that the fictitious æstimatio of the case - - - - $x^3 + c = bx$, always answers to the true root of - $x^3 = bx + c$. Cardan also shows what the relation of the coefficients is, when the case has no true roots, but only one fictitious root, which is the same as the true root of the reciprocal case, formed by changing the sign of the absolute number. Thus, the case $x^3 + 21 = 2x$ has no true root, and only one false root, viz, -3, which is the same as the true root of $x^3 = 2x + 21$: and he shows in general, that changing the sign of the absolute number in such cases as want the 2d term, or changing the signs of the even terms when it is not wanting, changes the signs of all the three roots, which he also illustrates by many examples; thus, the roots of $x^3 + 11x^2 = 72$, are $+\sqrt{40} - 4$, and -3, and $-\sqrt{40} - 4$; and the roots of $x^3 + 72 = 11x^2$, are $-\sqrt{40} + 4$, and $+3$, and $+\sqrt{40} + 4$.

And he further observes, that the sum of the three roots, or the difference between the true and fictitious roots, is equal to 11, the coefficient of the 2d term. He also shows how certain cubic cases have one, or two, or three roots, according to circumstances: that the case $x^4 + d = bx^2$ has sometimes four roots, and sometimes none at all, that is, no real ones: that the case $x^3 + bx = ax^2 + c$ may have three true æstimatios, or positive roots, but no fictitious or negative ones; and for this reason, that the odd powers of minus being minus, and the even powers plus, the two terms $x^3 + bx$ would be negative, and equal to a positive sum

$ax^2 + c$, which is absurd: and further, that the case $x^3 + ax^2 + bx = c$ has three roots, one plus and two minus, which are the same, with the signs changed, as the roots of the case $x^3 + bx + c = ax^2$. He also shows the relation of the coefficients when the equation has only one real root, in a variety of cases: but that the case $x^3 + c = ax^2 + bx$ has always one negative root, and either two positive roots, or none at all; the number of roots failing by pairs, or the impossible roots, as we now call them, being always in pairs. Of all these circumstances Cardan gives a great many particular examples in numeral coefficients, and subjoins geometrical demonstrations of the properties here enumerated; such as, that the two corresponding or reciprocal cases have the same root or roots, but with different signs or affections; and how many true or positive roots each case has.

Upon the whole, it appears from this short chapter, that Cardan had discovered most of the principal properties of the roots of equations, and could point out the number and nature of the roots, partly from the signs of the terms, and partly from the magnitude and relations of the coefficients. He shows in effect, that when the case has all its roots, or when none are impossible, the number of its positive roots is the same as the number of changes in the signs of the terms, when they are all brought to one side: that the coefficient of the 2d term is equal to the sum of all the roots positive and negative collected together, and consequently that when the 2d term is wanting, the positive roots are equal to the negative ones: and that the signs of all the roots are changed, by changing only the signs of the even terms: with many other remarks concerning the nature of equations.

In chap. 2, Cardan enumerates all the cases of compound equations of the 2d and 3d order, namely, 3 quadratics, and 19 cubics; with 44 derivatives of these two, that is, of the same kind, with higher denominations.

In chap. 3 are treated the roots of simple cases, or simple

equations, or at least that will reduce to such, having only two terms, the one equal to the other. He directs to depress the denominations equally, as much as they will, according to the height of the least; then divide by the number or coefficient of the greatest; and lastly extract the root on both sides. So, if $20x^3 = 180x^5$, then $20 = 180x^2$, and $\frac{1}{9} = x^2$, and $x = \frac{1}{3}$.

Chap. 4 treats of both general and particular roots, and contains various definitions and observations concerning them. It is here shown, that the several cases of quadratics and cubics have their roots of the following forms or kinds, namely, that the case

$x^2 = ax + b$ has its root of this kind $\sqrt{19} + 3$,
$x^2 + ax = b$ has its root of this kind $\sqrt{19} - 3$,
$x^2 + b = ax$ has two roots like $3 + \sqrt{2}$ and $3 - \sqrt{2}$,
$x^3 = bx + c$ has its root of this kind $\sqrt[3]{4} + \sqrt[3]{2}$,
$x^3 + bx = c$ has its root of this kind $\sqrt[3]{4} - \sqrt[3]{2}$,
$x^3 = ax^2 + c$ has this kind of root $\sqrt[3]{16} + 2 + \sqrt[3]{4}$,
$x^3 + ax^2 = c$ has this kind of root $\sqrt[3]{16} - 2 + \sqrt[3]{4}$,

where the three parts $\sqrt[3]{16}$, 2, $\sqrt[3]{4}$, are in continual proportion.

Chap. 5 treats of the æstimatio of the lowest degree of compound cases, that is, affected quadratic equations; giving the rule for each of the three cases, which consists in completing the square, &c, as at present, and which was the method given by the Arabians and Hindus; and proving them by geometrical demonstrations from Eucl. I, 43, and II, 4 and 5; in which he makes some improvement on the demonstrations of Mahomet. And hence it appears that the work of this Arabian author was in being, and well known in Cardan's time.

Chap. 6, on the methods of finding new rules, contains some curious speculations concerning the squares and cubes of binomial and residual quantities, and the proportions of the terms of which they consist, shown from geometrical demonstrations, with many curious remarks and properties,

forming a foundation of principles for investigating the rules for cubic equations.

Chap. 7 is on the transmutation of equations, showing how to change them from one form to another, by taking away certain terms out of them; as $x^3 + ax^2 = c$, to $x^3 = bx + d$, &c. The rules are demonstrated geometrically; and a table is added, of the forms into which any given cases will reduce; which transformations are extended to equations of the 4th and 5th order. And hence it appears that Cardan knew how to take away any term out of an equation.

Chap. 8 shows generally how to find the root of any such equation as this, $ax^m = x^n + b$, where m and n are any exponents whatever, but n the greater; and the rule is, to separate or divide the coefficient a into two such parts z and $a - z$, as that the absolute number b shall be equal to $(a - z) . z^{\frac{m}{n-m}}$, the product of the one part $a - z$, and the $\frac{m}{n-m}$ power of the other part: then the root x is $= z^{\frac{1}{n-m}}$. The rule is general for quadratics, cubics, and all the higher powers; and could not have been formed without the knowledge of the composition of the terms from the roots of the equation.

Chap. 9 and 10 contain the resolution of various questions producing equations not higher than quadratics.

Chap. 11 is on the case or form $x^3 + 3bx = 2c$. Cardan now comes to the actual resolution of the first case of cubic equations. He begins with relating a short history of the invention of it, observing that it was first found out, about thirty years before, by Scipio Ferreus, of Bononia, and by him taught to Antonio Maria Florido, of Venice, who having a contest afterwards with Nicolas Tartalea, of Brescia, it gave occasion to Tartalea to find it out himself, who after great entreaties taught it to Cardan, but suppressed the demonstration. By help of the rule alone, however, Cardan of himself discovered the source or geometrical investi-

gation, which he gives here at large, from Eucl. II, 4. In this process he makes use of the Greek letters α, β, γ, δ, &c, to denote certain indefinite numbers or quantities, to render the investigation general; which may be considered as the first instance of such literal notation in algebra. He then gives the rule in words at length, which comes to this,

$$x = \sqrt[3]{[\sqrt{(c^2 + b^3)} + c]} - \sqrt[3]{[\sqrt{(c^2 + b^3)} - c]};$$

illustrating it by a variety of examples; in the resolution of which, he extracts the cubic roots of such of the binomials as will admit of it, by some rule which he had for that purpose; such as $x^3 + 6x = 20$, where $x = \sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)} = (\sqrt{3} + 1) - (\sqrt{3} - 1) = 2$.

Chap. 12, on the case $x^3 = 3bx + 2c$. This he treats exactly as the last, and finds the rule to be

$$x = \sqrt[3]{[c + \sqrt{(c^2 - b^3)}]} + \sqrt[3]{[c - \sqrt{(c^2 - b^3)}]};$$

which he illustrates by many examples, as usual. But when b^3 exceeds c^2, which has since been called the irreducible case, he refers to another following book, called *Aliza*, for other rules of solution, to overcome this difficulty, about which he took infinite pains.

Chap. 13, on the case $x^3 + 2c = 3bx$. This case, by a geometrical process, he reduces to the case in the last chapter: thus, find the æstimatio y of the case $y^3 = 3by + 2c$, having the same coefficients as the given case $x^3 + 2c = 3bx$; then is $x = \frac{1}{2}y \pm \frac{1}{2}\sqrt{(12b - 3y^2)}$, giving two roots. He shows also how to find the second root, when the first is known, independent of the foregoing case. From this relation of these two cases he deduces several corollaries, one of which is, that the æstimatio or root of the case $y^3 = 3by + 2c$, is equal to the sum of the roots of the case $x^3 + 2c = 3bx$. As in the example $y^3 = 16y + 21$, whose æstimatio is $\sqrt{9\frac{1}{4}} + 1\frac{1}{2}$, which is equal to the sum of 3 and $\sqrt{9\frac{1}{4}} - 1\frac{1}{2}$, the two roots of the case $x^3 + 21 = 16x$.

In chapters 14, 15, and 16, he treats of the three cases which contain the 2d and 3d powers, but wanting the 1st

power, according to all the varieties of the signs; which he performs by exterminating the 2d term, or that which contains the 2d power of the unknown quantity x, by substituting $y \pm \frac{1}{3}$ the coefficient of that term for x, and so reducing these cases to one of the former. In these chapters Cardan sometimes also gives other rules; thus, for the case $x^3 + 3ax^2 = 2c$, find first the æstimatio y of the case $y^3 = 3ay + \sqrt{2c}$, then is $x = y^2 - 3a$: also for the case $x^3 + 2c = 3ax^2$, first find the two roots of $y^3 + 2c = 3ay\sqrt[3]{2c}$, then is $x = \frac{\sqrt[3]{4c^2}}{y}$ the two values of x according to the two values of y. He here also gives another rule, by which a second æstimatio or root is found, when the first is known, namely, if e be the first æstimatio, or value of x in the case $x^3 + 2c = 3ax^2$, then is the other value of

$$x = \tfrac{1}{2}\sqrt{[(3a - e) \cdot (3a + 3e)]} + \tfrac{1}{2}(3a - e).$$

In chapters 17, 18, 19, 20, 21, 22, 23, Cardan treats of the cases in which all the four terms of the equation are present; and this he always effects by taking away the 2d term out of the equation, and so reducing it to one of the foregoing cases which want that term, giving always geometrical investigations, and adding a great many examples of every case of the equations.

Chap. 24, is on the 44 derivative cases; which are only higher powers of the forms of quadratics and cubics.

Chap. 25, on imperfect and special cases; containing many particular examples when the coefficients have certain relations among them, with easy rules for finding the roots; also eight other rules for the irreducible case $x^3 = bx + c$.

Chap. 26, in like manner, contains easy rules for biquadratics, when the coefficients have certain special relations.

Then the following chapters, from chap. 27 to chap. 38, contain a great number of questions and applications of various kinds, the titles of which are these: *De transitu capituli specialis in capitulum speciale; De operationibus radi-*

eum pronicarum seu mixtarum et Allellarum; De regula modi; De regula Aurea; De regula Magna, or the method of finding out solutions to certain questions; *De regula æqualis positionis*, being a method of substituting for the half sum and half difference of two quantities, instead of the quantities themselves; *De regula inæqualiter ponendi, seu proportionis; De regula medii; De regula aggregati; De regula liberæ positionis; De regula falsum ponendi*, in which some quantities come out negative; *Quomodo excidant partes et denominationes multiplicando*. Among the foregoing collection of questions, which are chiefly about numbers, there are some geometrical ones, being the application of algebra to geometry; such as, In a right-angled triangle, given the sum of each leg and the adjacent segment of the hypotenuse, made by a perpendicular from the right angle, to determine the area, &c; with other such geometrical questions, resolved algebraically.

Chap. 39, *De regula qua pluribus positionibus invenimus ignotam quantitatem*; which is employed on biquadratic equations. After some examples of his own, Cardan gives a rule of Lewis Ferrari's, for resolving all biquadratics, namely, by means of a cubic equation, which Ferrari investigated at his request, and which Cardan here demonstrates, and applies in all its cases. The method is very general, and consists in forming three squares, thus: first, complete one side of the equation up to a square, by adding or subtracting some multiples or parts of some of its own terms on both sides, which it is always easy to do: 2d, supposing now the three terms of this square to be but one quantity, viz, the first term of another square to which this same side is to be completed, by annexing the square of a new and assumed indeterminate quantity, with double the product of the roots of both; which evidently forms the square of a binomial, consisting of the assumed indeterminate quantity and the root of the first square: 3d, the other side of the equation is then made to become the square of a binomial also, by supposing the product of its 1st and 3d

terms to be equal to the square of half its 2d term; for it consists of only three terms, or three different denominations of the original unknown quantity: then this equality will determine the value of the assumed indeterminate quantity, by means of a cubic equation, and from it, that of the original ignota, by the equal roots of the 2d and 3d squares. Here we have a notable example of the use of assuming a new indeterminate quantity to introduce into an equation, long before Des Cartes was born, who made use of a like assumption for a similar purpose. And this method is very general, and is here applied to all forms of biquadratics, either having all their terms, or wanting some of them. To illustrate this rule, I shall here set down the process in one of his examples, which is this, $x^4 + 4x + 8 = 10x^2$. Now first subtract $2x^2 + 4x + 7$ from both sides, then the first becomes a square, viz, $x^4 - 2x^2 + 1$ or $(x^2 - 1)^2 = 8x^2 - 4x - 7$. Next assume the indeterminate y, and subtract $2y(x^2 - 1) - y^2$ from both sides, making the first side again a square, viz, $(x - 1)^2 - 2y(x - 1) + y^2$ or $(x^2 - 1 - y)^2 = (8 - 2y)x^2 - 4x + y^2 + 2y - 7$. Of this latter side, make the product of the 1st and 3d terms equal to the square of half the 2d term, that is, $(8 - 2y) \cdot (y^2 + 2y - 7) = 2^2$, which reduces to $y^3 + 30 = 2y^2 + 15y$; the positive roots of which are $y = 2$ or $\sqrt{15}$; and hence, using 2 for y, the equation of equal squares becomes $(x^2 - 1 - y)^2$ or $(x^2 - 3)^2 = 4x^2 - 4x + 1$, the roots of which give $x^2 - 3 = 2x \backsim 1$; and hence $x^2 = 2x + 2$, or $x^2 + 2x = 4$; the two positive roots of which are $\sqrt{3} + 1$ and $\sqrt{5} - 1$, which are two of the values of x in the given equation $x^4 + 4x + 8 = 10x^2$. The other roots he leaves to be tried by the reader.

The 40th, or last, chap. is entitled, On modes of general supposition relating to this art; with some rules of an unusual kind; and æstimatios or roots of a nature different from the foregoing ones. Some of these are as follow: If $x^3 = ax^2 + c$, and $x - a = y$, and $x : y :: c : d$; then is $y^3 + ay^2 = d$.

Secondly, if $x^3 + ax^2 = c$,
and $y^3 = ay^2 + c$,
then is $x + a : y - a :: y^2 : x^2$.

Thirdly, when $x^3 + c = ax^2$, the square will be taken away, by putting $x = y + \frac{1}{3}a$; and then the equation becomes $y^3 + c - 2(\frac{1}{3}a)^3 = \frac{1}{3}a^2y$.

Cardan adds some other remarks concerning the solutions of certain cases and questions, all evincing the accuracy of his skill, and the extent of his practice; he then concludes the book with a remark concerning a certain transformation of equations, which quite astonishes us to find that the same person who, through the whole work, has shown such a profound and critical skill in the nature of equations, and the solution of problems, should yet be ignorant of one of the most obvious transmutations attending them, namely, increasing or diminishing the roots in any proportion. Cardan having observed that the form $x^3 = bx + c$ may be changed into another similar one, viz, $y^3 = \frac{b}{c}y + \sqrt{\frac{1}{c}}$, of which the coefficient of the term y is the quotient arising from the coefficient of x divided by the absolute number of the first equation; and that the absolute number of the 2d equation, is the root of the quotient of 1 divided by the said absolute number of the first; he then adds, that finding the æstimatio or root of the one equation from that of the other is very difficult, *valde difficilis*.

It is matter of wonder that Cardan, among so many transmutations, should never think of substituting, instead of x in such equations, another positio or root, greater or less than the former in any indefinite proportion, that is, multiplied or divided by a given number; for this would have led him immediately to the same transformation as he makes above, and that by a way which would have shown the constant proportion between the two roots. Thus, instead of x in the given form $x^3 = bx + c$, substitute dy, and it becomes $d^3y^3 = bdy + c$; and this divided by d^3 becomes $y^3 = \frac{b}{d^2}y + \frac{c}{d^3}$; and here if d be taken $= \sqrt{c}$, it becomes

$y^3 = \frac{b}{c} y + \sqrt{\frac{1}{c}}$; which is the transformation in question, and in which it is evident that x is $= y \sqrt{c}$, and $y = \frac{x}{\sqrt{c}}$. Instead of this, Cardan gives the following strange way of finding the one root x from the other y, when this latter is by any means known; viz, Multiply the first given equation by $y^2 x + 1$, then add $\frac{x^2}{4y^2}$ to both sides, and lastly extract the roots of both, which can always be done, as they will always be both of them squares; and the roots will give the value of x by a quadratic equation.

Thus, $x^3 = bx + c$ multiplied by $y^2 x + 1$ gives $y^2 x^4 + x^3 = by^2 x^2 + (cy^2 + b)\, x + c$; add $\frac{x^2}{4y^2}$, then $y^2 x^4 + x^3 + \frac{x^2}{4y^2} = by^2 + \frac{1}{4y^2} \cdot x^2 + (b + cy^2)\, x + c$; and the roots are $yx^2 + \frac{x}{2y} = \sqrt{[(by^2 + \frac{1}{4y^2})x^2 + (b + cy^2)\, x + c]}$; and this 2d side of the equation, he says, will always have a root also. It is indeed true, that it will have an exact root; but the reason of it is not obvious, which is, because y is the root of the equation $y^3 = \frac{b}{c} y + \sqrt{\frac{1}{c}}$. Cardan has not shown the reason why this happens; but I apprehend he made it out in this manner, viz, similar to the way in which he forms the last square in the case of biquadratic equations, namely, by making the product of the 1st and 3d terms equal to the square of half the 2d term: thus, in the present case, it is $4c\,(by^2 + \frac{1}{4y^2}) = (b + cy^2)^2$, which reduces to $y^3 = \frac{b}{c} y + \sqrt{\frac{1}{c}}$ the equation in question. Therefore taking y the root of the equation $y^3 = \frac{b}{c} y + \sqrt{\frac{1}{c}}$, and substituting its value in the quantity $(by^2 + \frac{1}{4y^2})\, x^2 + (b + cy^2)\, x + c$, this becomes a complete square.

Of CARDAN's *Libellus de Aliza Regula.*

Subjoined to the above treatise on cubic equations, is this *Libellus de Aliza Regula*, or the algebraic logistics, in which the author treats of some of the more abstruse parts of arithmetic and algebra, especially cubic equations, with many more attempts on the irreducible case $x^3 = bx + c$. This book is divided into 60 chapters; but I shall only set down the titles of some few of them, whose contents require more particular notice.

Chap. 4. *De modo redigendi quantitates omnes, quæ dicuntur latera prima ex decimo Euclidis in compendium.* He treats here of all Euclid's irrational lines, as surd numbers, and performs various operations with them.

Chap. 5. *De consideratione binomiorum et recisorum, &c; ubi de æstimatione capitulorum.* Contains various operations of multiplying compound numbers and surds.

Chap. 6. *De operationibus* p: *et* m: (i. e. + and −) *secundum communem usum.* Here it is shown that, in multiplication and division, *plus* always gives the same signs, and *minus* gives the contrary signs. So also in addition, every quantity retains its own sign; but in subtraction they change the signs. That the $\sqrt{+}$, or the square root of plus, is +; but the $\sqrt{-}$, or the square root of minus, is nothing as to common use: (but of this below). That $\sqrt[3]{-}$ is −; as $\sqrt[3]{-8}$ is -2. That a residual, composed of + and − may have a root also composed of + and −: so $\sqrt{(5-\sqrt{24})}$ is $=\sqrt{3}-\sqrt{2}$. The rules for the signs in multiplication and division are illustrated by this example; to divide 8 by $2+\sqrt{6}$ or $\sqrt{6}+2$. Take the two corresponding residuals $2-\sqrt{6}$ and $\sqrt{6}-2$, and by these multiply both the divisor and dividend; then the products are + and − respectively, and the quotients still both alike. Thus,

Divid.	Divis.	Divid.	Divis.
8	$\sqrt{6}+2$	8	$2+\sqrt{6}$
$\sqrt{6}-2$	$\sqrt{6}-2$	$2-\sqrt{6}$	$2-\sqrt{6}$
$\sqrt{384}-16$ divide	$+2$	$16-\sqrt{384}$ div.	-2
Quot. $\sqrt{96}-8$.		Quot. $\sqrt{96}-8$.	

And this method of performing division of compound surds, was fully taught before him, by Lucas de Burgo, namely, reducing the compound divisor to a simple quantity, by multiplying by the corresponding quantity, having the sign changed.

In chap. 11 and 18, and elsewhere, Cardan makes a general notation of a, b, c, d, e, f, for any indefinite quantities, and treats of them in a general way.

Cap. 2. *De contemplatione* p: *et* m: (or $+$ and $-$), *et quod* m: *in* m: *facit* p: *et de causis horum juxta veritatem.* Cardan here demonstrates geometrically that, in multiplication and division, like signs give plus, and unlike signs give minus. And he illustrates this numerically, by squaring the quantity 8, or $6 + 2$, or $10 - 2$, which must all produce the same thing, namely, 64.

Among many of the chapters which treat of the irreducible case $x^3 = bx + c$, there is a peculiar kind of way given in chap. 31, which is entitled *De æstimatione generali* $x^3 = bx + c$ *solida vocata, et operationibus ejus;* in which he shows how to approximate to the root of that case, in a manner similar to approximating the square root and cube root of a number. The rule he uses for this purpose, is the 3d in chap. 25 of the last book, and it is this: Divide b into two parts, such that the sum of the products of each, multiplied by the square root of the other, may be equal to $\frac{1}{2}c$; then the sum of the roots of these parts is the æstimatio or value of x required. So, of this equation $x^3 = 10x + 24$; the two parts are 9 and 1, and their roots 3 and 1, and their sum $4 = x$, as in the margin.

$$\begin{array}{cc} 9 & 1 \\ \multicolumn{2}{c}{\times} \\ 3 & 1 \end{array}$$

$3 + 9$ or $12 = \frac{1}{2}c$,

$x = 3 + 1 = 4$.

Again, take $x^3 = 6x + 1$. Here he invents a new notation to express the root or *radix*, which he calls *solida*, viz, $x = \surd$ *solida* 6 *in* $\frac{1}{2}$, that is, the roots of the two parts of 6, so that each part multiplied by the root of the other, the two products may be $\frac{1}{2}$ or $\frac{1}{2}c$. Then to free this from fractions, and make the operation easier, multiply that root by some number, as suppose 4, that is, the square part 6 by

the square of 4, and the solid part $\frac{1}{2}$ by the cube of 4; then $x = \frac{1}{4}\sqrt{}$ *solida* 96 *in* 32. Now, by a few trials, it is found that the parts are

nearly $95\frac{8}{9}$ and $\frac{1}{9}$, which give too much,
or $95\frac{9}{10}$ and $\frac{1}{10}$, which give too little,

and thereof $95\frac{17}{19}$ and $\frac{2}{19}$ are still nearer. Divide both by 4^2 or 16, then $5\frac{151}{152}$ and $\frac{1}{152}$ are the quotients. And the sum of their roots, or $x = \sqrt{5\frac{151}{152}} + \sqrt{\frac{1}{152}}$ is nearly the value of the root x.

Cap. 42. *De duplici æquatione comparanda in capitulo cubi et numeri æqualium rebus.* Treats of the two positive roots of that case, neglecting the negative one; and showing, not only that the case has two such roots, but that the same number may be the common root of innumerable equations.

Cap. 57. *De tractatione æstimationis generalis capituli* $x^3 = bx + c$. Cardan here again resumes the consideration of the irreducible case, making ingenious observations upon it, but still without obtaining the root by a general rule. In this place also, as well as elsewhere, he shows how to form an equation in this case, that shall have a given binomial root, as suppose $\sqrt{m} + n$, where the equation will be $x^3 = (m + 3n^2)x + 2n(m - n^2)$, having $\sqrt{m} + n$ for one root, namely the positive root. From which it appears that he was well acquainted with the composition of cubic equations from given roots.

Cap. 59. *De ordine et exemplis in binomiis secundo et quinto.* Contains a great many numeral forms of the same irreducible case $x^3 = bx + c$, with their roots; from which are derived these following cases, with many curious remarks. As, when

$$x^3 = (c + 1)x + c, \text{ then } x = \sqrt{(c + \tfrac{1}{4})} + \tfrac{1}{2}.$$
$$x^3 = (\tfrac{1}{2}c + 4)x + c, \text{ then } x = \sqrt{(\tfrac{1}{2}c + \tfrac{4}{4})} + \tfrac{2}{2}.$$
$$x^3 = (\tfrac{1}{3}c + 9)x + c, \text{ then } x = \sqrt{(\tfrac{1}{3}c + \tfrac{9}{4})} + \tfrac{3}{2}.$$
$$x^3 = (\tfrac{1}{4}c + 16)x + c, \text{ then } x = \sqrt{(\tfrac{1}{4}c + \tfrac{16}{4})} + \tfrac{4}{2}.$$
$$x^3 = (\frac{1}{n}c + n^2)x + c, \text{ then } x = \sqrt{(\frac{1}{n}c + \frac{n^2}{4})} + \frac{n}{2}.$$

Cap. 60. *Demonstratio generalis capituli cubi æqualis rebus et numero.* This demonstration of the irreducible case is geometrical, like all the rest. Some more ingenious remarks are again added, as if he reluctantly finished the book, without perfectly overcoming the difficulty of the irreducible case. Cardan here also uses the letters a and b for any two indefinite numbers, in order to show the form and manner of the arithmetical operations: thus $\frac{a}{b}$ is the fraction for their quotient, also $\sqrt{\frac{a}{b}}$ or $\frac{\sqrt{a}}{\sqrt{b}}$ the square root of that quotient, and $\sqrt[3]{\frac{a}{b}}$ or $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ the cube root of it, &c.

Having considered the chief contents of Cardan's algebra, it will now be proper to sum them up, and set down a list of the improvements made by him, as collected from his writings:

And 1st, Tartalea having only communicated to him the rules for resolving these three cases of cubic equations, viz,

$$\left.\begin{array}{l} x^3 + bx = c, \\ x^3 = bx + c, \\ x^3 + c = bx, \end{array}\right\}$$

he thence raised a very large and complete work, laying down rules for all forms and varieties of cubic equations, having all their terms, or wanting any of them, and having all possible varieties of signs; demonstrating all these rules geometrically; and treating very fully of almost all sorts of transformations of equations, in a manner before unknown.

2nd, It appears that he was well acquainted with all the roots of equations that are real, both positive and negative; or, as he calls them, true and fictitious; and that he made use of them both occasionally. He also showed, that the even roots of positive quantities, are either positive or negative; that the odd roots of negative quantities, are real and negative; but that the even roots of them are impossible, or nothing as to common use. He was also acquainted with the

3d Number and nature of the roots of an equation, and that partly from the signs of the terms, and partly from

the magnitude and relation of the coefficients. He also knew,

4th, That the number of positive roots is equal to the number of changes of the signs of the terms.

5th, That the coefficient of the second term of the equation, is the difference between the positive and negative roots.

6th, That when the second term is wanting, the sum of the negative roots is equal to the sum of the positive roots.

7th, How to compose equations that shall have given roots.

8th, That, changing the signs of the even terms, changes the signs of all the roots.

9th, That the number of roots failed in pairs; or what we now call impossible roots were always in pairs.

10th, To change the equation from one form to another, by taking away any term out of it.

11th, To increase or diminish the roots by a given quantity. It appears also,

12th, That he had a rule for extracting the cube root of such binomials as admit of extraction.

13th, That he often used the literal notation *a*, *b*, *c*, *d*, &c.

14th, That he gave a rule for biquadratic equations, suiting all their cases; and that, in the investigation of that rule, he made use of an assumed indeterminate quantity, and afterwards found its value by the arbitrary assumption of a relation between the terms.

15th, That he applied algebra to the resolution of geometrical problems. And

16th, That he was well acquainted with the difficulty of what is called the irreducible case, viz, $x^3 = bx + c$, upon which he spent a great deal of time, in attempting to overcome it. And though he did not fully succeed in this case, any more than other persons have done since, he nevertheless made many ingenious observations about it, laying down

rules for many particular forms of it, and showing how to approximate very nearly to the root in all cases whatever.

Cardan died at Rome in the year 1575.

OF TARTALEA.

Nicholas Tartalea, or Tartaglia, of Brescia, was contemporary with Cardan, and was probably older than he was; but I do not know of any book of algebra published by him till the year 1546, the year after the date of Cardan's work on Cubic Equations, when he printed his *Quesiti et Inventioni diverse*, at Venice, where he resided as a public lecturer on mathematics. This work is dedicated to our king Henry the 8th of England, and consists of nine books, containing answers to various questions which had been proposed to him at different times, concerning mechanics, statics, hydrostatics, &c; but it is only the 9th, or last book, that we shall have occasion to take notice of in this place, as it contains all those questions which relate to arithmetic and algebra. These are all set down in chronological order, forming a pretty collection of questions and solutions on those subjects, with a short account of the occasion of each of them. Among these, the correspondence between him and Cardan forms a remarkable part, as we have here the history of the invention of the rules for cubic equations, which he communicated to Cardan, under the promise, and indeed oath, to keep them secret, on the 25th of March, 1539. But, notwithstanding his oath, finding that Cardan published them in 1545, as above related, it seems Tartalea published the correspondence between them, in revenge for his breach of faith; and it elsewhere appears, that many other sharp bickerings passed between them on the same account, which only ended with the death of Tartalea, in the year 1557.

It seems it was a common practice among the mathematicians, and others, of that time, to send to each other curious and difficult questions, as trials of skill; and to this

circumstance it is that we owe the principal questions and discoveries in this collection, as well as many of the best discoveries of other authors. The collection now before us contains questions and solutions, with their dates, in a regular order, from the year 1521, and ending in 1541, in forty-two dialogues, the last of which is with an English gentleman, namely, Mr. Richard Wentworth, who it seems was no mean mathematician, and who learned some algebra, &c, of Tartalea, while he resided at Venice. The questions at first are mostly very easy ones in arithmetic, but gradually become more difficult, and exercising simple and quadratic equations, with complex calculations of radical quantities: all showing that he was well skilled in the art of algebra as it then stood, and that he was very ingenious in applying it to the solution of questions. Tartalea made no alteration in the notation or forms of expression used by Lucas de Burgo, calling the first power of the unknown quantity, in his language, *cosa*, the second power *censa*, the third *cubo*, &c, and writing the names of all the operations in words at length, without using any contractions, except the initial ℞ for root or radicality. So that the only thing remarkable in this collection, is the discovery of the rules for cubic equations, with the curious circumstances attending the same.

The first two of these were discovered by Tartalea in the year 1530, namely, for the two cases $x^3 + ax^2 = c$, and $x^3 = ax^2 + c$, as appears by Quest. 14 and 25 of this collection, on occasion of a question then proposed to him by one Zuanni de Tonini da Coi or Colle, John Hill, who kept a school at Brescia. And from the 25th question we learn, that he discovered the rules for the other two cases $x^3 + bx = c$, and $x^3 = bx + c$, on the 12th and 13th of February, 1535, at Venice, where he had come to reside the year before. And the occasion of it was this: There was then at Venice one Antonio Maria Fiore or Florido, who, by his own account, had received from his preceptor Scipio Farreo, about thirty years before, a general rule for resolving the case

$x^3 + bx = c$. Being a captious man, and presuming on this discovery, Florido used to brave his contemporaries, and by his insults provoked Tartalea to enter into a wager with him, that each should propose to the other thirty different questions; and that he who soonest resolved those of his adversary, should win from him as many treats for himself and friends. These questions were to be proposed on a certain day at some weeks distance; and Tartalea made such good use of his time, that eight days before the time appointed for delivering the propositions, he discovered the rules both for the case $x^3 + bx = c$, and the case $x^3 = bx + c$. He therefore proposed several of his questions, so as to fall either on this latter case, or on the cases of the cube and square, expecting that his adversary would propose his in the former. And what he suspected fell out accordingly; the consequence of which was, that on the day of meeting Tartalea resolved all his adversary's questions in the space of two hours, without receiving one answer from Florido in return; to whom, however, Tartalea generously remitted the forfeit of the thirty treats won of him.

Question 31 first brings us acquainted with the correspondence between Tartalea and Cardan. This correspondence is very curious, and would well deserve to be given at full length in their own words, if it were not too long for this place. I may enlarge farther upon it under the article *Cubic Equations*; but must here be content with a brief abstract only. Cardan was then a respectable physician, and lecturer in mathematics, at Milan; who having nearly finished the printing of a large work on arithmetic, algebra, and geometry, and having heard of Tartalea's discoveries in cubic equations, he was very desirous of drawing those rules from him, that he might add them to his book before it was finished. For this purpose, he first applied to Tartalea, by means of a third person, a bookseller, whom he sent to him, in the beginning of the year 1539, with many flattering compliments, and offers of his services and friendship, &c, accompanied with some critical questions for him to resolve,

according to the custom of the times. Tartalea, however, refused to disclose his rules to any one; as the knowledge of them gained him great reputation among all people, and gave him a great advantage over his competitors for fame, who were commonly afraid of him on account of those very rules. He only sent Cardan, therefore, at his request, a copy of the thirty questions which had been proposed to him in the contest with Florido.

Not to be rebuffed so easily, Cardan next applied, in the most urgent manner, by letter, to Tartalea; which, however, procured from him only the solution of some other questions proposed by Cardan, with a few of the questions that had been proposed to Florido, but none of their solutions. Finding he could not thus prevail, with all his fair promises, Cardan then fell upon another scheme. There was at Milan a certain Marquis dal Vasto, a great patron of Cardan, and, it was said, of learned men in general. Cardan conceived the idea of making use of the influence of this nobleman to draw Tartalea to Milan, hoping that then, by personal intreaties, he should succeed in drawing the long-concealed rules from him. Accordingly he wrote a second letter to Tartalea, much in the same strain with the former, strongly inviting him to come and spend a few days in his house at Milan, and representing that, having often commended him in the highest terms to the marquis, this nobleman desired much to see him; for which reason Cardan advised him, as a friend, to come to visit them at Milan, as it might be greatly to his interest, the marquis being very liberal and bountiful; and he besides gave Tartalea to understand, that it might be dangerous to offend such a man by refusing to come, who might, in that case, take offence, and do him some injury. This manœuvre had the desired effect: Tartalea on this occasion laments to himself in these words, "By this I am reduced to a great dilemma; for if I go not to Milan, the marquis may take it amiss, and some evil may befal me on that account; I shall therefore go, though very unwillingly." When he arrived at Milan, however, the

marquis was gone to Vigeveno; and Tartalea was prevailed on to stay three days with Cardan, in expectation of the marquis returning; at the end of which time he set out from Milan, with a letter from Cardan, to go to Vigeveno to that nobleman. While Tartalea was at Milan the three days, Cardan plied him by all possible means, to draw from him the rules for the cubic equations; and at length, just as Tartalea was about to depart from Milan, on the 25th of March, 1539, he was overcome by the most solemn protestations of secrecy that could be made. Cardan says, "I shall swear to you on the holy evangelists, and by the honour of a gentleman, not only never to publish your inventions, if you reveal them to me; but I also promise to you, and pledge my faith as a true Christian, to note them down in cyphers, so that after my death no other person may be able to understand them." To this Tartalea replies, "If I refuse to give credit to these assurances, I should deservedly be accounted utterly void of belief. But as I intend to ride to Vigeveno, to see his excellency the marquis, as I have been here now these three days, and am weary of waiting so long; whenever I return, therefore, I promise to show you the whole." Cardan answers, "Since you determine at any rate to go to Vigeveno to the marquis, I shall give you a letter for his excellency, that he may know who you are. But now before you depart, I intreat you to show me the rule for the equations, as you have promised." "I am content," says Tartalea: "But you must know, that to be able on all occasions to remember such operations, I have brought the rule into rhyme; for if I had not used that precaution, I should often have forgot it; and though my rhymes are not very good, I do not value that, as it is sufficient that they serve to bring the rule to mind as often as I repeat them. I shall here write the rule with my own hand, that you may be sure I give you the discovery exactly." These rude verses contain, in rather dark and enigmatical language, the rule for these three cases, viz,

$\left.\begin{array}{l} x^3 + bx = c, \\ x^3 = bx + c, \\ x^3 + c = bx, \end{array}\right\}$ which differ, however, only in the sign of one quantity, and the rule amounts to this: Find two numbers, z and y, such that their difference in the first case, and their sum in the 2d and 3d, may be equal to c the absolute number, and their product equal to the cube of $\frac{1}{3}$ of b the coefficient of the less power; then the difference of their cube roots will be equal to x in the first case, and the sum of their cube roots equal to x in the 2d and 3d cases: that is, taking $z - y = c$ in the 1st case, or $z + y = c$ in the 2d and 3d, and $zy = (\frac{1}{3}b)^3$; then $x = \sqrt[3]{z} - \sqrt[3]{y}$ in the first case, and $x = \sqrt[3]{z} + \sqrt[3]{y}$ in the other two. At parting, Tartalea fails not again to remind Cardan of his obligation: "Now your excellency will remember not to break your promised faith, for if unhappily you should insert these rules either in the work you are now printing, or in any other, though you should even give them under my name, and as of my invention, I promise and swear that I shall immediately print another work that will not be very pleasing to you." "Doubt not," says Cardan, "but that I shall observe what I have promised: Go, and rest secure as to that point; and give this letter of mine to the marquis." It should seem, however, that Tartalea was much displeased at having suffered himself to be worried as it were out of his rules; for as soon as he quitted Milan, instead of going to wait upon the marquis, he turned his horse's head, and rode straight home to Venice, saying to himself, "By my faith I shall not go to Vigeveno, but shall return to Venice, come of it what will."

After Tartalea's departure, it seems Cardan applied himself immediately to resolving some examples in the cubic equations by the new rules, but not succeeding in them; for indeed he had mistaken the words, as it was very easy to do in such bad verses, having mistaken $(\frac{1}{3}b)^3$ for $\frac{1}{3}b^3$, or the cube of $\frac{1}{3}$ of the coefficient, for $\frac{1}{3}$ of the cube of the coefficient; accordingly we find him writing to Tartalea in fourteen days after the above, blaming him much for his abrupt departure without seeing the marquis, who was so

liberal a prince, he said, and requesting Tartalea to resolve him the example $x^3 + 3x = 10$. This Tartalea did to his satisfaction, rightly guessing at the nature of his mistake; and concludes his answer with these emphatical words, "Remember your promise." On the 12th of May following, Cardan returns him a letter of thanks, together with a copy of his book, saying, "As to my work, just finished, to remove your suspicion, I send you a copy, but unbound, as it is yet too fresh to be beaten. But as to the doubt you express lest I may print your inventions, my faith, which I gave you with an oath, should satisfy you; for as to the finishing of my book, that could be no security, as I could always add to it whenever I please. But on account of the dignity of the thing, I excuse you for not relying on that which you ought to have done, namely, on the faith of a gentleman, instead of the finishing of a book, which might at any time be enlarged by the addition of new chapters; and there are besides a thousand other ways. But the security consists in this, that there is no greater treachery than to break one's faith, and to aggrieve those who have given us pleasure. And when you shall try me, you will find whether I be your friend or not, and whether I shall make an ungrateful return for your friendship, and the satisfaction you have given me."

It was within less than two months after this, however, that Tartalea received the alarming news of Cardan's showing some symptoms of breaking the faith he had so lately pledged to him: this was in a letter from a quondam pupil of his, in which he writes, "A friend of mine at Milan has written to me, that Dr. Cardano is composing another algebraical work, concerning some lately-discovered rules; hence I imagine they may be those same rules which you told me you had taught him; so that I fear he will deceive you." To which Tartalea replies, "I am heartily grieved at the news you inform me of, concerning Dr. Cardano of Milan; for if it be true, they can be no other rules but those I gave him; and therefore the proverb truly says,

'That which you wish not to be known, tell to nobody.' Pray endeavour to learn more of this matter, and inform me of it."

Tartalea, after this, kept on the reserve with Cardan, not answering several letters he sent him, till one written on the 4th of August the same year, 1539, complaining greatly of Tartalea's neglect of him, and farther requesting his assistance to clear up the difficulty of the irreducible case $x^3 = bx + c$, which Cardan had thus early been embarrassed with: he says that when $(\frac{1}{3}b)^3$ exceeds $(\frac{1}{2}c)^2$, the rule cannot be applied to the equation in hand, because of the square root of the negative quantities. On this occasion Tartalea turns the tables on Cardan, and plays his own game back upon him; for being aware of the above difficulty, and unable to overcome it himself, he wanted to try if Cardan could be encouraged to accomplish it, by pretending that the case might be done, though in another way. He says thus to himself, "I have a good mind to give no answer to this letter, no more than to the other two. However I will answer it, if it be but to let him know what I have been told of him. And as I perceive that a suspicion has arisen concerning the difficulty or obstacle in the rule for the case $x^3 = bx + c$, I have a mind to try if he can alter the data in hand, so as to remove the said obstacle, and to change the rule into another form, though I believe indeed that it cannot be done; however there is no harm in trying."—"M Hieronime, I have received your letter, in which you write that you understand the rule for the case $x^3 = bx + c$; but that when $(\frac{1}{3}b)^3$ exceeds $(\frac{1}{2}c)^2$, you cannot resolve the equation by following the rule, and therefore you request me to give you the solution of this equation $x^3 = 9x + 10$. To which I reply, that you have not used a good method in that case, and that your whole process is intirely false. And as to resolving you the equation you have sent, I must say that I am very sorry that I have already given you so much as I have done; for I have been informed, by a credible person, that you are about to publish another algebraical work,

and that you have been boasting through Milan of having discovered some new rules in algebra. But, take notice, that if you break your faith with me, I shall certainly keep my word with you, nay, I even assure you to do more than I promised." In Cardan's answer to this he says, "You have been misinformed as to my intention to publish more on algebra. But I suppose you have heard something about my work *De Mysteriis Æternitatis*, which you take for some algebra I intend to publish. As to your repenting of having given me your rules, I am not to be moved from the faith I promised you for any thing you say." To this, and many other things contained in the same letter, Tartalea returned no answer, being still suspicious of Cardan's intentions, and declining any more correspondence with him.

This however did not discourage C. for we find him writing again to T. on the 5th of January, 1540, to clear up another difficulty which had occurred in this business, namely to extract the cube root of the binomials, of which the two parts of the rule always consisted, and for which purpose it seems C. had not yet found out a rule. On this occasion he informs T. that his quondam competitor Zuanne Colle had come to Milan, where, in some contests between them, Colle gave Cardan to understand that he had found out the rules for the two cases $x^3 + bx = c$, and $x^3 = bx + c$, and further that he had discovered a general rule for extracting the cube roots of all such binomials as can be extracted; and that, in particular, the cube root of $\sqrt{108} + 10$ is $\sqrt{3} + 1$, and that of $\sqrt{108} - 10$ is $\sqrt{3} - 1$, and consequently that $\sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}$ is $= (\sqrt{3} + 1) - (\sqrt{3} - 1) = 2$. He then earnestly entreats T. to try to find out the rule, and the solution of certain other questions which had been proposed to him by Colle. By this letter T. is still more confirmed in his resolutions of silence; so that, without returning any answer, he only sets down among his own memorandums some curious remarks on the contents of the letter, and then concludes to himself, "Wherefore I do not choose to answer him again, as I

have no more affection for him than for M. Zuanne, and therefore I shall leave the matter between them." Among those remarks he sets down a rule for extracting the cube root of such binomials as can be extracted, and that is done from either member of the binomial alone, thus: Take either term of the binomial, and divide it into two such parts that one of them may be a complete cube, and the other part exactly divisible by 3; then the cube root of the said cubic part will be one term of the required root, and the square root of the quotient arising from the division of $\frac{1}{3}$ of the 2d part by the cube root of the first, will be the other member of the root sought. This rule will be better understood in characters thus: let m be one member of the given binomial, whose cube root is sought; and let it be divided into the two parts a^3 and $3b$, so that $a^3 + 3b$ be $= m$; then is $a + \sqrt{\frac{b}{a}}$ the cube root required, if it has one. Thus, in the quantity $\sqrt{108} + 10$, taking the term 10 for m, then 10 divides into 1 and 9, where $a^3 = 1$ or $a = 1$, and $3b = 9$ or $b = 3$: therefore $a + \sqrt{\frac{b}{a}}$ becomes $1 + \sqrt{3}$ for the cube root of $\sqrt{108} + 10$. And taking the other member $\sqrt{108}$, this divides into the two equal parts $\sqrt{27}$ and $\sqrt{27}$, making $a^3 = \sqrt{27}$, and $3b = \sqrt{27}$; hence $a = \sqrt{3}$, and $b = \sqrt{3}$ also; consequently $a + \sqrt{\frac{b}{a}}$ is $= \sqrt{3} + \sqrt{\frac{3}{3}}$ or $\sqrt{3} + 1$ for the cube root of the binomial sought, the same as before. "And thus, he adds, we may know whether any proposed binomial or risidual be a cube or a noncube; for if it be a cube, the same two terms for the root must arise from both the given terms separately; and if the two terms of the root cannot thus be brought to agree both ways, such binomial or risidual will not be a cube." And thus ends the correspondence between them, at least for this time. But it seems they had still more violent disputes when C. in violation of his faith, so often pledged to the contrary, published his work on cubic equations 4 years afterwards, viz, in the year 1545, of which we have before

given an account, which disputes, it appears, continued till the death of Tartalea in the year 1557.

The last article in the volume contains a dialogue on some other forms of cubic equations, in the year 1541, between T. and Mr. Richard Wentworth, the English gentleman before-mentioned, who it seems had resided some time at Venice, on some public service from England, as T. in the dedication of the volume to Henry VIII. king of England, makes mention of him as "a gentleman of his sacred majesty." Mr. Wentworth had learned some mathematics of T. and being about to depart for England, requests T. to show him his newly discovered rules for cubic equations, as a farewell-lesson; and it is worth while to note a few particulars in this conference, as they show pretty well the limited knowledge of T. at that time, as to the nature and roots of such equations. T. had before, it seems, showed Mr. W. the rules for the cases of the 3d and 1st powers, and now the latter desires him to do the same as to the three cases in which the 3d and 2d powers only are concerned. On this T. professes great gratitude to Mr. W. for many obligations, but desires to be excused from giving him the rules for these, because he says he intends soon to compose a new work on Arithmetic, Geometry, and Algebra, which he intends to dedicate to him, and in which he means to insert all his new discoveries. On Mr. W. urging him further, however, T. gives him the roots of some equations of that kind, as for instance:

If $x^3 + 6x^2 = 100$, then

$$x = \sqrt[3]{(42 + \sqrt{17000})} + \sqrt[3]{(42 - \sqrt{17000})} - 2.$$

If $x^3 + 9x^2 = 100$, then $x = \sqrt{24} - 2$.

If $x^3 + 3x^2 = 2$, then $x = \sqrt{3} - 1$.

If $x^3 + 4 = 5x^2$, then $x = \sqrt{8} + 2$.

If $x^3 + 6 = 7x^2$, then $x = \sqrt{15} + 3$.

But he does not mention the rules for finding them.

In the course of the conversation T. tells him that "all such equations admit of two different answers, and perhaps more; and hence it follows that they have, or admit of,

two different rules, and perhaps more, the one more difficult than the other." And on Mr. W. expressing his wonder at this circumstance of a plurality of roots, T. replies, "It is however very true, though hardly to be believed, and indeed if experience had not confirmed it, I should scarcely have believed it myself." He then commits a strange blunder in an example which he takes to illustrate this by, namely the equation $x^3 + 3x = 14$, which, he says, it is evident has the number 2 for one of its roots; and yet, he adds, "whoever shall resolve the same equation by my rule, will find the value of x to be $\sqrt[3]{(7 + \sqrt{50})} + \sqrt[3]{(7 - \sqrt{50})}$, which is proved to be a true root by substituting it in the equation for x. And therefore, continues he, it is manifest that the case $x^3 + bx = c$ admits of two rules, namely, one (as in the above example) which ought to give the value of x rational, viz, 2, and the other is my rule, which gives the value of x irrational, as appears above; and there is reason to think that there may be such a rule as will give the value of $x = 2$, though our ancestors may not have found it out." ——"And these two different answers will be found not only in every equation of this form $x^3 + bx = c$, when the value of x happens to be rational, as in the example $x^3 + 3x = 14$ above, but the same will also happen in all the other five forms of cubic equations: and therefore there is reason to think that they also admit of two different rules; and by certain circumstances attending some of them, I am almost certain that they admit of more than two rules, as, God willing, I shall soon demonstrate."

Now all this discourse shows a strange mixture of knowledge and ignorance: it is very probable that he had met with some equations which admit of a plurality of roots; indeed it was hardly possible for him to avoid it: but it seems he had no suspicion what the number of roots might be, nor that his reasoning in this instance was founded on an error of his own, mistaking the root $x = \sqrt[3]{(7 + \sqrt{50})} + \sqrt[3]{(7 - \sqrt{50})}$, of the equation $x^3 + 3x = 14$, for a different root from the number or root 2, when in reality it is

the very same, as he might easily have found, if he had extracted the cube roots of the binomials by the rule which he himself had just given above for that purpose: for by that rule he would have found $\sqrt[3]{(7 + \sqrt{50})} = 1 + \sqrt{2}$, and $\sqrt[3]{(7 - \sqrt{50})} = 1 - \sqrt{2}$, and therefore their sum is $2 = x$, the same root as the other, which T. thought had been different. And besides this root 2, the equation in hand, $x^3 + 3x = 14$, admits of no other real roots. Nor indeed does any equation of the same form, $x^3 + bx = c$, admit of more than one real root.

It seems also they had not yet discovered that all cases belong to the rules and forms for quadratic equations, which have only two powers in them, in which the exponent of the one is just double of the exponent of the other, as $x^{2n} + bx^n = c$; but some particular cases only of this sort they had as yet ventured to refer to quadratics, as the case $x^4 + bx^2 = c$. But, in the conclusion of this dialogue, T. informs W. of another case of this sort which he had accomplished, as a notable discovery, in these words: "I well remember, says he, that in the year 1536, on the night of St. Martin, which was on a Saturday, meditating in bed when I could not sleep, I discovered the general rule for the case $x^6 + bx^3 = c$, and also for the other two, its accompanying cases, in the same night." And then he directs that they are to be resolved like quadratics, by completing the square, &c. And in these resolutions it is remarkable that he uses only the positive roots, without taking any notice of the negative ones.

Tartalea also published in Venice, in 1556, &c, a very large work, in folio, on Arithmetic, Geometry, and Algebra. This is a very complete and curious work on the first two branches; but that of Algebra is carried no farther than quadratic equations, called *book the first*, with which the work terminates. It is evidently incomplete, owing to the death of the author, which happened before this latter part of the work was printed, as appears by the dates, and by the prefaces. It appears also, from several parts of this

work, that the author had many severe conflicts with Cardan and his friend Lewis Ferrari: and particularly, there was a public trial of skill between them, in the year 1547; in which it would seem that Tartalea had greatly the advantage, his questions mostly remaining unanswered by his antagonists.

OF MICHAEL STIFELIUS.

After the foregoing analysis of the works of the first algebraic writers in Italy, it will now be proper to consider those of their contemporaries in Germany; where it is remarkable that, excepting for the discoveries in cubic equations, the art was in a more advanced state, and of a form approaching nearer to that of our modern Algebra; the state and circumstances indeed being so different, that one would almost be led to suppose they had derived their knowledge of it from a different origin.

Here Stifelius and Scheubelius were writers of the same time with Cardan and Tartalea, and even before their discoveries, or publication, concerning the rules for cubic equations, Stifelius's *Arithmetica Integra* was published at Norimberg in 1544, being the year before Cardan's work on cubic equations, and is an excellent treatise, both on Arithmetic and Algebra. The work is divided into three books, and is prefaced with an Introduction by the famous Melanchthon. The first book contains a complete and ample Treatise on Arithmetic, the second an Exposition of the 10th book of Euclid's Elements, and the third a Treatise on Algebra, being therefore properly the part with which we are at present concerned. In the dedication of this part, he ascribes the invention of Algebra to Geber, an Arabic Astronomer; and mentions besides, the authors Campanus, Christ. Rudolph, and Adam Ris, Risen, or Gigas, whose rules and examples he has chiefly given. In other parts of the book he speaks, and makes use also, of the works of Boetius, Campanus, Cardan (i. e. his Arithmetic published

in 1539, before the work on cubic equations appeared), de Cusa, Euclid, Jordan, Milichius, Schonerus, and Stapulensis. So that he appears to have been very little acquainted indeed with any besides the German authors.

Chap. 1. *On the Rule of Algebra, and its parts.* Stifelius here describes the notation and marks of powers or denominations as he calls them, which marks for the several powers are thus:

1st,	2d,	3d,	4th,	5th,	6th,	&c.
𝔯,	𝔷,	𝔠,	𝔷𝔷,	ss,	𝔷𝔠,	&c.

being formed from the initials of the barbarous way in which the Germans pronounced and wrote the Latin and Italic names of the powers, namely, res or cosa, zensus, cubo, zensi-zensus, sursolid, zensi-cubo, &c. And the coss or first power 𝔯, he calls the radix or root, which is the first time that we meet with this word in the printed authors. He also here uses the signs or characters, + and −, for addition and subtraction, and the first of any that I know of: for in Italy they used none of these characters for a long time after. He has no mark however for equality, but makes use of the word itself.

Chap. 2. On the Parts of the Rule of Geber or Algebra: teaching the various reductions by addition, subtraction, multiplication, division, involution, and evolution, &c.

Chap. 3. On the Algorithm of Cossic numbers; teaching the usual operations of addition, subtraction, multiplication, division, involution, and extraction of roots, much the same as they are at present. Single terms, or powers, he calls simple quantities; but such as 1 𝔷 + 1 𝔯 a composite or compound, and 2 𝔯 − 8 a defective one. In multiplication and division, he proves that like signs give +, and unlike signs −. He shows that the powers 1, 𝔯, 𝔷, 𝔠, &c, form a geometrical progression from unity; and that the natural series of numbers 0, 1, 2, 3, &c, from 0, are the exponents of the cossic powers; and he, for the first time, expressly calls them exponents: thus,

Exponents,	0,	1,	2,	3,	4,	5,	6,	&c.
Powers,	1,	℞,	ʒ,	¢,	ʒʒ,	*ss*,	ʒ¢,	&c.

And he shows the use of the exponents, in multiplication, division, powers, and roots, as we do at present; viz, adding the exponents in multiplication, and subtracting them in division, &c. And these operations he demonstrates from the nature of arithmetical and geometrical progressions. It is remarkable that these compound denominations of the powers are formed from the simple ones according to the *products* of the exponents, while those of Diophantus are formed according to the *sums* of them; thus the 6th power here is ʒ¢ or quadrato-cubi, but with Diophantus it is cubo-cubi; and so of others. Which is presumptive evidence that the Europeans had not taken their Algebra immediately from him, independent of other proofs.

Chap. 4. On the extraction of the roots of cossic numbers. He here treats of quadratic equations, which he resolves by completing the square, from Euclid II. 4, &c. Also quadratics of the higher orders, showing how to resolve them in all cases, whatever the height may be, provided the exponents be but in arithmetical progression, as

$\left.\begin{array}{l}2,\ 1,\ 0\\ 4,\ 2,\ 0\\ 6,\ 3,\ 0\\ 8,\ 4,\ 0\end{array}\right\}$ &c; where it is plain that he always counts 0 for the exponent of the unknown quantity in the absolute term.

Chap. 5. Of irrational cossic numbers, and of surd or negative numbers. In this treatise of radicals, or irrationals, he first uses the character √ to denote a root, and sets after it the mark of the power whose root is intended; as √ʒ20 for the square root of 20, and √¢20 for the cube root of the same, and so on. He treats here also of negative numbers, or what he calls surd or fictitious, or numbers less than 0. On which he takes occasion to observe, that when a geometrical progression is continued downwards below 1, then the exponents of the terms, or the arithmetical progression, will go below 0 into negative numbers,

and will yet be the true exponents of the former; as in these,

Expon.	−3	−2	−1	0	1	2	3
Pow.	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

And he gives examples to show that these negative exponents perform their office the same as the positive ones, in all the operations.

Chap. 6. On the perfection of the Rule of Algebra, and of Secondary Roots. In the reduction of equations, he uses a more general rule than those who had preceded him, who detailed the rule in a multitude of cases; instead of which, he directs to multiply or divide the two sides equally, to transpose the terms with + or −, and lastly to extract such root as may be denoted by the exponent of the highest power.

As to secondary roots, Cardan treated of a 2d *ignota* or unknown, which he called *quantitas*, and denoted it by the initial *q*, to distinguish it from the first. But here Stifelius, for distinction sake, and to prevent one root from being mistaken for others, assigns literal marks to all of them, as A, B, C, D, &c, and then performs all the usual operations with them, joining them together as we do now, except that he subjoins the initial of the power, instead of its numeral exponent: thus,

3A into 9B makes 27AB,
3ʒ into 4B makes 12ʒB,
2ȼ into 4Aʒ makes 8ȼA,
1A squared makes 1Aʒ,
6 into 3C makes 18C,
2Aʒ into 5Aȼ makes 10A*ss*, &c, &c.
8ȼAʒ divided by 4ȼ makes 2Aʒ, &c.
The square root of 25Aʒ is 5A, &c.
Also 2A added to 2℞ makes 2℞ + 2A,
and 2A subtr. from 2℞ makes 2℞ − 2A.

And he shows how to use the same, in questions concerning several unknown numbers; where he puts a different character for each of them, as ℞, A, B, C, &c; he then makes

out, from the conditions of the question, as many equations as there are characters; from these he finds the value of each letter, in terms of some one of the rest; and so, expelling them all but that one, reduces the whole to a final equation, as we do at present.

The remainder of the book is employed with the solutions of a great number of questions, to exercise all the rules and methods; some of which are geometrical ones.

From this account of the state of Algebra in Stifelius, it appears that the improvements made by himself, or other Germans, beyond those of the Italians, as contained in Cardan's book of 1539, were as follow:

1st. He introduced the characters +, −, √, for plus, minus, and root, or *radix*, as he calls it.

2d. The initials 𝓇, ʒ, 𝒸, &c. for the powers.

3d. He treated all the higher orders of quadratics by the same general rule.

4th. He introduced the numeral exponents of the powers, −3, −2, −1, 0, 1, 2, 3, &c, both positive and negative, so far as integral numbers, but not fractional ones; calling them by the name *exponens*, exponent: and he taught the general uses of the exponents, in the several operations of powers, as we now use them, or the logarithms.

5th. And lastly, he used the general literal notation A, B, C, D, &c, for so many different unknown or general quantities.

OF SCHEUBELIUS.

John Scheubelius published several books on Arithmetic, and Algebra. The one now before me, is intitled *Algebræ Compendiosa Facilisque Descriptio, quâ depromuntur magna Arithmetices miracula. Authore Johanne Scheubelio Mathematicarum Professore in Academia Tubingensi. Parisiis* 1552. But at the end of the book it is dated 1551. The work is most beautifully printed, and is a very clear though succinct treatise; and both in the form and matter much

resembles a modern printed book. He says that the writers ascribe this art to Diophantus, which is the first time that I find this Greek author mentioned by the modern algebraists: he further observes, that the Latins call it *Regula Rei & Census*, the rule of the thing and the square, or of the 1st and 2d power; and the Arabs, Algebra. His characters and operations are much the same as those of Stifelius, using the signs and characters $+$, $-$, $\surd$, and the powers ℞, , ʒ, ȼ, &c, where the character ℞ is used for 1 or unity, or a number, or the 0 power; prefixing also the numerical coefficients; thus 44sʒ + 11ʒ + 31 ℞ − 53ℊ. He uses also the exponents 0, 1, 2 3, &c, of the powers, the same way as Stifelius, before him. He performs the algebraical calculations, first in integers, and then in fractions, much the same as we do at present. Then of equations, which he says may be of infinite degrees, though he treats only of two, namely the first and second orders, or what we call simple and quadratic equations, in the usual way, taking however only the positive roots of these; and adverting to all the higher orders of quadratics, namely,

$$x^4,\ ax^2,\ b;$$
$$x^6,\ ax^3,\ b;$$
$$x^8,\ ax^4,\ b;\ \&c.$$

Next follows a tract on surds, both simple and compound, quadratic, cubic, binomial, and residual. Here he first marks the notation, observing that the root is either denoted by the initial of the word, or, after some authors, by the mark $\surd$:, viz. the sq. root $\surd$:, the cube root w$\surd$:, and the 4th root, or root of the root thus v$\surd$:, which latter method he mostly uses. He then gives the Arithmetic of surds, in multiplication, division, addition, and subtraction. In these last two rules he squares the sum or difference of the surds, and then sets the root to the whole compound, which he calls *radix collecti*, what Cardan calls *radix universalis*. Thus $\surd 12 \pm \surd 20$ is ra. col. $32 \pm \surd 960$. But when the terms will reduce to a common surd, he then unites them into one number; as $\surd 27 + \surd 12$ is equal $\surd 75$. Also

of cubic surds, and 4th roots. In binomial and residual surds, he remarks the different kinds of them which answer to the several irrational lines in the 10th book of Euclid's elements; and then gives this general rule for extracting the root of any binomial or residual $a \pm b$, where one or both parts are surds, and a the greater quantity, namely, that the square root of it is $\sqrt{\frac{a + \sqrt{(a^2 - b^2)}}{2}} \pm \sqrt{\frac{a - \sqrt{(a^2 - b^2)}}{2}}$; which he illustrates by many examples. This rule will only succeed however, so as to come out in simple terms, in certain cases, namely, either when $a^2 - b^2$ is a square, or when a and $\sqrt{(a^2 - b^2)}$ will reduce to a common surd, and unite: in all other cases the root is in two compound surds, instead of one. He gives also another rule, which comes however to the same thing as the former, though by the words of them they seem to be different.

Scheubelius wrote much about the time of Cardan and Stifelius. And as he takes no notice of cubic equations, it is probable he had neither seen nor heard any thing about them; which might very well happen, the one living in Italy, and the other in Germany. And, besides, I know not if this be the first edition of Scheubel's book: it is rather likely it is not, as it is printed at Paris, and he himself was professor of mathematics at Tubingen in Germany.

ROBERT RECORDE.

To this ingenious man we are indebted for the first treatise on algebra, then named the Cossic Art, in the English language; but his meritorious labours, like those of the greatest benefactors of mankind, appear to have been ill requited, since, after removing to the capital, he died under confinement for debt in the Fleet-prison. In his book on Arithmetic he is stiled "teacher of mathematics and practitioner in physic at Cambridge." It was for many ages the custom to unite the title as well as the practice of medicine with those of chemistry, alchymy, mathematics, and astrology,

by the Moors, and after them by the Europeans, and is still continued among the almanac-makers. And it is remarkable, that as the Moors were not less famed in Europe for their skill in medicine, than their dexterity in calculation, the term Physician and Algebraist appear at first to have been regarded as almost synonimous. Thus, it is curious to remark, that in the celebrated romance of Don Quixote, published about this time, the bachelor Samson Carrasco, who in his rencounter with the knight, was thrown from his horse, and had his ribs broken, sent in quest of an *Algebrista* to heal his bruises.

The first part of his Arithmetic was published in 1552; and the second part in 1557, under the title of "The Whetstone of Witte, which is the seconde parte of Arithmetike: containing the Extraction of Rootes: The Cossike Practise, with the Rule of Equation: and the Workes of Surde Nombers." The work is in dialogue between the master and scholar; and is nearly after the manner of the Germans, Stifelius and Scheubelius, but especially the latter, whom he often quotes, and takes examples from. The chief parts of the work are, 1st, The properties of abstract and figurate numbers. 2d, The extraction of the square and cube roots, much the same as at present. Here, when the number is not an exact power, but having some remainder over, he either continues the root into decimals as far as he pleases, by adding to the remainders always periods of cyphers; or else makes a vulgar fraction for the remaining part of the root, by taking the remainder for the numerator, and double the root for the denominator, in the square root; but in the cube root he takes, for the denominator, either the triple square of the root, which is Cardan's rule, or the triple square and triple root, with one more, which is Scheubel's rule. 3d, Of Algebra, or "Cossike Nombers." He uses the notation of powers with their exponents the same as Stifel, with all the operations in simple and compound quantities, or integers and fractions. And he gives also many examples of extracting the roots of

compound algebraic quantities, even when the roots are from two to six terms, in imitation of the same process in numbers, just as we do at present; which is the first instance of this kind that I have observed. As of this quantity:

Square Root.

25ʒ ʒ𝔠𝔢 + 80ſʒ − 26ʒ ʒ − 63ʒ(5 𝔠𝔢 + 8ʒ − 9 ℈.

4th, " The Rule of Equation, commonly called Algeber's Rule." He here, first of any, introduces the character =, for brevity sake. His words are, " And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or gemowe lines of one lengthe, thus: =, bicause noe 2 thynges can be moare equalle." He gives the rules for simple and quadratic equations, with many examples. He gives also some examples in higher compound equations, with a root for each of them, but gives no rule how to find it. 5th, " Of Surde Nombers." This is a very ample treatise on surds, both simple and compound, and surds of various degrees, as square, cubic, and biquadratic, marking the roots in Scheubel's manner, thus: $\surd$, w$\surd$, v$\surd$. He here uses the names bimedial, binomial, and residual; but says they have been used by others before him, though this is the first place where I have observed the two latter.—Hence it appears that the things which chiefly are new in this author, are these three, viz.

1. The extraction of the roots of compound algebraic quantities.
2. The use of the terms binomial and residual.
3. The use of the sign of equality, or =.

OF PELETARIUS.

The first edition of this author's algebra was printed in 4to at Paris, in 1558, under this title, *Jacobi Peletarii Cenomani, de occulta parte Numerorum, quam Algebram vocant. Lib. duo.*

In the preface he speaks of the supposed authors of Algebra, namely Geber, Mahomet the son of Moses, an Arabian, and Diophantus. But he thinks the art older, and mentions some of his contemporary writers, or a very little before him, as Cardan, Stifel, Scheubel, Chr. Januarius; and a little earlier again, Lucas Paciolus of Florence, and Stephen Villafrancus a Gaul.

Of the two books, into which the work is divided, the first is on rational, and the second on irrational or surd quantities; each being divided into many chapters. It will be sufficient to mention only the principal articles.

He calls the series of powers *numeri creati*, or derived numbers, or also radicals, because they are all raised from one root or *radix*. He names them thus, radix, quadratus cubus, quadrato-quadratus, or biquadratus, supersolidus, quadrato-cubus, &c; and marks them thus ℞, *q*, *ſſ*, *qq*, *ss*, *qſſ*, *bss*, &c. Of these he gives the following series in numbers, having the common ratio 2, with their marks set over them, and the exponents set over these again, in an arithmetical series, beginning at 0, thus:

0	1	2	3	4	5	6	7	8	
1	℞	*q*	*ſſ*	*qq*	*ss*	*qſſ*	*bss*	*qqq*	
1	2	4	8	16	32	64	128	256	&c.

And he shows the use of the exponents, the same as Stifelius and Scheubelius; like them also he prefixes coefficients to quantities of all kinds, as also the radical $\surd$. But he does not follow them in the use of the signs + and −, but employs the initials *p* and *m* for the same purpose. After the operations of addition, &c, he performs involution, and evolution also, much the same way as at present: thus, in powers, raise the coefficient to the power required, and multiply the exponent, or sign, as he calls it, by 2, or 3, or 4, &c, for the 2nd, 3d, 4th, &c, power; and the reverse for extraction: and hence he observes, if the number or coefficient will not exactly extract, or the sign do not exactly divide, the quantity is a surd.

After the operations of compound quantities, and fractions, and reduction of equations, namely, simple and quadratic equations, as usual, in chap. 16, *De Inveniendis generatim Radicibus Denominatorum*, he gives a method of finding the roots of equations among the divisors of the absolute number, when the root is rational, whether it be integral or fractional; for then, he observes, the root always lies hid in that number, and is some one of its divisors. This is exemplified in several instances, both of quadratic and cubic equations, and both for integral and fractional roots. And he here observes, that he knows not of any person who has yet given general rules for the solution of cubic equations; which shows that when he wrote this book, either Cardan's last book was not published, or else it had not yet come to his knowledge.

Chap. 17 contains, in a few words, directions for bringing questions to equations, and for reducing these. He here observes, that some authors call the unknown number *res*, and others the *positio*; but that he calls it *radix*, or root, and marks it thus ℞: hence the term, root of an equation. But it was before called radix by Stifelius.

Chap. 21 & *seq.* treat of secondary roots, or a plurality of roots, denoted by A, B, C, &c, after Stifelius.

The 2d book contains the like operations in surds, or irrational numbers, and is a very complete work on this subject indeed. He treats first of simple or single surds, then of binomial surds, and lastly of trinomial surds. He gives here the same rule for extracting the root of a binomial and residual as Scheubelius, viz, $\sqrt{(a \pm b)} = \sqrt{\frac{a + \sqrt{(a^2 - b^2)}}{2}} \pm \sqrt{\frac{a - \sqrt{(a^2 - b^2)}}{2}}$. In dividing by a binomial or residual, he proceeds as all others before him had done, namely, reducing the divisor to a simple quantity, by multiplying it by the same two terms with the sign of one of them changed, that is by the binomial if it be a residual, or by the same residual if it be a binomial; and multiplying the

dividend by the same thing: thus

$$\frac{3}{\sqrt{5}-2}=\frac{3}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{3\sqrt{5}+6}{5-4}=3\sqrt{5}+6.$$

And, in imitation of this method, in division by trinomial surds, he directs to reduce the trinomial divisor first to a binomial or residual, by multiplying it by the same trinomial with the sign of one term changed, and then to reduce this binomial or residual to a simple nominal as above; observing to multiply the dividend by the same quantities as the divisor. Thus, if the divisor be $4+\sqrt{2}-\sqrt{3}$; multiplying this by $4+\sqrt{2}+\sqrt{3}$, the product is $15+8\sqrt{2}$; then this binomial multiplied by the residual $15-8\sqrt{2}$, gives $225-128$ or 97 for the simple divisor: and the dividend, whatever it is, must also be multiplied by the two $4+\sqrt{2}+\sqrt{3}$ and $15-8\sqrt{2}$. Or in general, if the divisor be $a+\sqrt{b}-\sqrt{c}$; multiply it by $a+\sqrt{b}+\sqrt{c}$, which gives $(a+\sqrt{b})^2-c=a^2+b-c+2a\sqrt{b}$; then multiply this by $a^2+b-c-2a\sqrt{b}$, and it gives - $(a^2+b-c)^2-4a^2b$, which will be rational, and will all collect into one single term. But Tartalea must have been in possession of some such rule as this, as one of the questions he proposed to Florido was of this nature, namely to find such a quantity as multiplied by a given trinomial surd, shall make it rational: and it appears, from what is done above, that, the given trinomial being $a+\sqrt{b}-\sqrt{c}$, the answer will be $(a+\sqrt{b}+\sqrt{c})\times(a^2+b-c+2\sqrt{b})$.

Chap. 24 shows the composition of the cube of a binomial or residual, and thence remarks on the root of the case or equation 1 : *p* 3℞ equal to 10, which he seems to know something about, though he had not Cardan's rules.

Chap. 30, which is the last, treats of certain precepts relating to square and cubic numbers, with a table of such squares and cubes for all numbers to 140; also showing

how to compute them both, by adding always their differences.

He then concludes with remarking that there are many curious properties of these numbers, one of which is this, that the sum of any number of the cubes, taken from the beginning, always makes a square number, the root of which is the sum of the roots of the cubes; so that the series of squares so formed, have for their

roots - - - - 1, 3, 6, 10, 15, 21, &c.
whose diff. are the natural nos. 1, 2, 3, 4, 5, 6, &c.

Namely, $1^3 = 1^2$; $1^3 + 2^3 = 3^2$; $1^3 + 2^3 + 3^3 = 6^2$, &c.

Or in general, $1^3 + 2^3 + 3^3 \cdots n^3 = (1 + 2 + 3 \cdots n)^2 = [\frac{1}{2}n.\ (n + 1)]^2 = \frac{1}{4}n^2 (n + 1)^2$.

This work of Peletarius is a very ingenious and masterly composition, treating in an able manner of the several parts of the subject then known, excepting the cubic equations. But his real discoveries, or improvements, may be reduced to these three, viz.

1st. That the root of an equation, is one of the divisors of the absolute term.

2d. He taught how to reduce trinomials to simple terms, by multiplying them by compound factors.

3d. He taught curious precepts and properties concerning square and cube numbers, and the method of constructing a series of each by addition only, namely by adding successively their several orders of differences.

PETER RAMUS.

Peter Ramus wrote his arithmetic and algebra about the year 1560. His notation of the powers is thus, *l*, *q*, *c*, *bq*, being the initials of latus, quadratus, cubus, biquadratus. He treats only of simple and quadratic equations. And the only thing remarkable in his work, is the first article, on the names and invention of Algebra, which we have noticed at the beginning of this history.

OF PEDRO NUGNEZ OR NUÑEZ, OR IN LATIN NONIUS.

Peter Nunez or Nonius, was a very ingenious and eminent physician and mathematician, for the time in which he lived. He was born in 1497, at Alcazar in Portugal, and died in 1577, at 80 years of age. He was professor of mathematics in the university of Coimbra, where he published several ingenious and useful pieces on different branches of the mathematics, as may be seen by the account of his life given in my Dictionary; but it is only with his Algebra that our business is at present.

This work he had composed in Portuguese, but translated it into the Castilian tongue, when he resolved on making it public, which he thought would render his book more useful, as this language was more generally known than the former. The dedication, to his former pupil, prince Henry, was dated from Lisbon, Dec. 1, 1564; and the work contains 341 leaves, equal to 682 closely printed pages, in the Antwerp edition of 1567, in 8vo; the folios being numbered only on one side.

The work is very methodically and plainly treated; being divided into regular and distinct chapters or sections; leading the reader gradually through the several operations of computation, in integers and fractions, in powers and roots, in surds and in proportions, &c. The rules for dignities or powers are given; and these he denominates from the product of their indices; thus, for the powers of 2, with their names, and denominations under them:

2	4	8	16	32	64	
Co.	Ce.	Cu.	Ce. Ce.	Re. p°.	Ce. Cu. or Cu. Ce.	&c.
1	2	3	4	5	6	

where the 6th denomination is called Ce. Cu. or Cu. Ce. that is Censo-Cubo or Cubo-Censo, meaning the square cubed or the cube squared, the index 6 denoted by 2×3, the product of the indices of the powers; after the manner of the former European authors, Lucas de Burgo, Tartalea,

and Cardan. And exactly after the manner of these also is his practice in every other part, with little or no variation, as far as he goes, which is to quadratic equations; without treating on Cubics, further than giving some account of the dispute between Tartalea and Cardan concerning their invention; and that in such a manner as shows he did not very well understand them. Like those authors are his names and rules for Raizes, roots or radicals, which he sometimes calls Sorda, surds. In their marks or signs also; as *R* for root; $\tilde{p}$ for $+$; $\tilde{m}$ for minus; *R. u* for root universal, instead of the vinculi used by the moderns; also *L* for ligature or composition, such as *L. R.* 9 with *R* 4, composed of 3 which is *R* 9, and of 2 which is *R* 4, making 5. Places likewise the name of the root after the *R*; as *R. cu.* 25. $\tilde{m}$. *R. cu.* 15. $\tilde{p}$. *R. cu.* 9; that is, $\sqrt[3]{25} - \sqrt[3]{15} + \sqrt[3]{9}$.

After all the usual preparatory rules, Nunez then treats of equations, simple and quadratic, in the common way, and giving geometrical demonstrations of the rules, as had been done before. He then applies these equations in the solution of a great number of examples, of questions or problems, first in numbers or arithmetic, and then in geometrical problems or figures; in which he proceeds orderly through the several kinds; as squares, rectangles, triangles, rhombs, rhomboids, trapeziums, pentagons, &c: all which he calls by the same names as at present. The content of a figure too he calls its Area, as De Burgo did before him; and after the same author also he gives the geometrical demonstration of the common rule for finding the area of a triangle from the three sides given. He also treats on the inscription of circles and squares in triangles of various kinds; and the division of triangles into several parts in different ways.

In an address to his readers, at the end of the book, Nunez informs them what are the authors whose books, on this subject, are to be found in Spain, which consist only of De Burgo, Tartalea, and Cardan; stating his ideas on the merits of their works, with critical remarks on many

parts of them, in which he more particularly approves of those of Tartalea. Upon the whole, the merit of Nunez, in this art, consists chiefly, or wholly, in having given a very neat and orderly treatise on it, after the manner of those authors; but without having made any improvements or inventions of his own, in the art.

On this occasion it is very remarkable to observe, that Nunez appears not to have been at all acquainted with any of the Germanic authors, several of whom were contemporaries of Tartalea and Cardan, and who treated the subject in a better manner, in some respects, than these did. Another thing may be here noticed on this occasion, as remarkable; not only that we have never heard of any other early writers on this subject, in Spain or Portugal; but that we have never heard of writings on it by the Moors, who occupied great part of that peninsula during several centuries, by whom we have always been taught to believe that the arts of arithmetic and algebra were brought into that country.—Had that people left any such works in the country, would not some of them have been found in some of the great cities or the universities, or could they yet be in existence there?—Or could the people have carried all their books away with them when they were expelled from the country?

BOMBELLI.

Raphael Bombelli's Algebra was published at Bologna in the year 1572, in the Italian language. In a short, but neat, introduction, he first adverts, in a few words, to the great excellence and usefulness of arithmetic and algebra. He then laments that it had hitherto been treated in so imperfect and irregular a way; and declares it his intention to remedy all defects, and to make the science and practice of it as easy and perfect as may be. And for this purpose he first resolved to procure and study all the former authors. He then mentions several of these, with a short history

or character of them; as Mahomet the son of Moses, an Arabian; Leonard Pisano; Lucas de Burgo, the first printed author in Europe; Oroncius; Scribelius; Boglione Francesi; Stifelius in Germany; a certain Spaniard, perhaps meaning Nunez or Nonius; and lastly Cardan, Ferrari, and Tartalea; with some others since, whose names he omits. He then adds a curious paragraph concerning Diophantus: he says that some years since there had been found, in the Vatican library, a Greek work on this art, composed by a certain Diophantus, of Alexandria, a Greek author, who lived in the time of Antoninus Pius; which work having been shown to him by Mr. Antonio Maria Pazzi Reggiano, public lecturer on mathematics at Rome; and finding it to be a good work, these two formed the design of giving it to the world; and he says that they had already translated five books, of the seven which were then extant, being as yet hindered by other avocations from completing the work. He then adds the following strange circumstance, viz. *that they had found that in the said work the Indian authors are often cited; by which he learned that this science was known among the Indians before the Arabians had it:* a paragraph the more remarkable as I have never understood that any other person could ever find, in Diophantus, any reference to Indian writers: and I have examined his work with some attention, for that purpose. Probably the copy which Bombelli saw, contained marginal remarks by Planudes, or some other scholiast, making mention of the Indian works on the science, or some such remarks; which might be mistaken for part of the text of Diophantus.

Bombelli's work is divided into three books. In the first, are laid down the definitions and operations of powers and roots, with various kinds of radicals, simple and compound, binomial, residual, &c; mostly after the rules and manner of former writers, excepting in some few instances, which I shall here take notice of. And first of his rule for the cube root of binomials or residuals, which for the sake of brevity, may be expressed in modern notation as follows: let $\sqrt{b}$

$+ a$ be the binomial, the term $\sqrt{b}$ being greater than a; then the rule for the cube root of $\sqrt{b} + a$ comes to this, $P - Q + \sqrt{[(P - Q)^2 + \sqrt[3]{(b - a)^2}]}$; where $P = \sqrt[3]{(\sqrt{(\frac{a^2}{64} + \frac{b - a^2}{64}} + \frac{a}{8})}$, and $Q = \sqrt[3]{(\sqrt{(\frac{a^2}{64} + \frac{b - a^2}{64})} - \frac{a}{8})}$. Which is a rule that can be of little or no use. For, in the first place, $(P - Q)^2 + \sqrt[3]{(b - a^2)}$ is the same as $(P + Q)^2$; and P or $\sqrt[3]{(\sqrt{(\frac{a^2}{64} + \frac{b - a^2}{64})} + \frac{a}{8})}$ is $= \sqrt[3]{(\sqrt{\frac{b}{64}} + \frac{a}{8})} = \sqrt[3]{\frac{\sqrt{b} + a}{8}} = \frac{1}{2} \sqrt[3]{(\sqrt{b} + a)}$; therefore the whole $P - Q + \sqrt{(P - Q)^2 + \sqrt[3]{(b - a^2)}}$ reduces to $P - Q + P + Q = 2P = 2 \times \frac{1}{2}\sqrt[3]{(\sqrt{b} + a)} = \sqrt[3]{(\sqrt{b} + a)}$, the original quantity first proposed.

The next thing remarkable in this 1st book, is his method for the square roots of negative quantities, and his rule for the cube roots of such imaginary binomials as arise from the irreducible case in cubic equations. His words, translated, are these: "I have found another sort of cubic root, very different from the former, which arises from the case of the cube equal to the first power and a number, when the cube of the $\frac{1}{3}$d part of the (coef. of the) 1st power, is greater than the square of half the absolute number, which sort of square root hath, in its algorism, names and operations different from the others; for in that case, the excess cannot be called either plus or minus; I therefore call it *plus of minus* when it is to be added, and *minus of minus* when it is to be subtracted." He then gives a set of rules for the signs when such roots are multiplied, and illustrates them by a great many examples. His rule for the cube roots of such binomials, viz, such as $a + \sqrt{-b}$, is this: First find $\sqrt[3]{(a^2 + b)}$; then, by trials search out a number c, and a sq. root $\sqrt{d}$, such, that the sum of their squares $c^2 + d$ may be $= \sqrt[3]{(a^2 + b)}$, and also $c^3 - 3cd = a$; then shall $c + \sqrt{-d}$ be $= \sqrt[3]{(a + \sqrt{-b})}$ sought. Thus, to extract the cube root of $2 + \sqrt{-121}$: here $\sqrt[3]{(a^2 + b)} = \sqrt[3]{125} = 5$; then taking $c = 2$, and $d = 1$, it is $c^2 + d = 5 = \sqrt[3]{(a^2 + b)}$, and $c^3 - 3cd = 8 - 6 = 2 = a$, as it ought; and therefore $2 + \sqrt{-1}$ is $=$ the cube root of $2 + \sqrt{-121}$, as required.

The notation in this book, is the initial *R* for root, with *q* or *c*, &c after it, for quadrate or cubic, &c root. Also *p* for *plus*, and *m* for *minus*.

In the 2d book, Bombelli treats of the algorism with unknown quantities, and the resolution of equations. He first gives the definitions and characters of the unknown quantity and its powers; in which he deviates from the former authors, but professes to imitate Diophantus. He calls the unknown quantity *tanto*, and marks it
thus - - - - $\overset{1}{\smile}$,
Its square or 2d power *potenza*, $\overset{2}{\smile}$,
Its cube - - - *cubo*, $\overset{3}{\smile}$,
and the higher names are compounded of these, and marked $\overset{4}{\smile}$, $\overset{5}{\smile}$, $\overset{6}{\smile}$, $\overset{7}{\smile}$, &c; so that he denotes all the powers by their exponents set over the common character $\smile$. And all these powers he calls by the general name *dignita*, dignity. He then performs all the algorism of these powers, by means of their exponents, as we do at present, viz, adding them in multiplication, subtracting in division, multiplying them by the index in involution, and dividing by the same in evolution.

In equations he goes regularly through all the cases, and varieties of the signs and terms; first all the simple or single powers, and then all the compound cases; demonstrating the rules geometrically, and illustrating them by many examples.

In compound quadratics, he gives two rules: the first is by freeing the *potenza* or square from its coefficient by division, and then completing the square, &c, in the usual way: and the 2d rule, when the first term has its coefficient, may be thus expressed; if $ax^2 + bx = c$, then $x = \frac{\sqrt{(ac + \frac{1}{4}b^2)} - \frac{1}{2}b}{a}$, being a way that was practised by the Indians. He takes only the positive root or roots; and in the case $ax^2 + c = bx$, which has two, he observes that the nature of the problem must show which of the two is the proper one.

In the cubic equations, he gives the rules and transformations, &c, after the manner of Cardan; remarking that some of the cases have only one root, but others two or three, of which some are true, and others false or negative. And in one place he says that by means of the case $x^3 = bx + c$ he *trisects* or *divides an angle into three equal parts*.

When he arrives at biquadratic equations, and particularly to this case $x^4 + ax = b$, he says, "Since I have seen Diophantus's work, I have always been of opinion that his chief intention was to come to this equation, because I observe he labours at finding always square numbers, and such, that adding some number to them, may make squares; and I believe that the six books, which are lost, may treat of this equation, &c."—"But Lewis Ferrari," he adds, "of this city, also laboured in this way, and found out a rule for such cases, which was a very fine invention, and therefore I shall here treat of it the best I can." This he accordingly does, in all the cases of biquadratics, both with respect to the number of terms in the equation, and the signs of the terms, except I think this most general case only $rx - qx^2 + px^3 - x^4 = s$; fully applying Ferrari's method in all cases. Which concludes the 2d book.

The 3d book consists only of the resolution of near 300 practical questions, as exercises in all the rules and equations, many of which are taken from other authors, especially from the first five books of Diophantus.

Upon the whole it appears, that this is a plain, explicit, and very orderly treatise on algebra, in which are very well explained the rules and methods of former writers. But Bombelli does not produce much of improvement or invention of his own, except his notation, which varies from others, and is by means of one general character, with the numeral indices of Stifelius. He also first remarks that angles are trisected by a cubic equation.

CLAVIUS.

Christopher Clavius wrote his Algebra about the year 1580, though it was not published till 1608, at Orleans. He mostly follows Stifelius and Scheubelius in his notation and method, &c, having scarcely any variations from them; nor does he treat of cubic equations. He mentions the names given to the art, and the opinions about its origin, in which he inclines to ascribe it to Diophantus, from what Diophantus says in his preface to Dyonisius.

STEVINUS.

The Arithmetic of Simon Stevin of Bruges, was published in 1585, and the same with his Algebra in 1605, in the Flemish dialect, and containing a free translation of the first 4 books of Diophantus. They were also printed in a French edition of his works at Leyden in 1634, with some notes and additions of Albert Girard, who it seems died the year before, this edition being published for the benefit of Girard's widow and children. This edition contains all the 6 books of Diophantus.

The Algebra is an ingenious and original work. He denotes the *res*, or unknown quantity, in a way of his own, namely by a small circle $\bigcirc$, within which he places the numeral exponent of the power, as $⓪$, $①$, $②$, $③$, &c. which are the 0, 1, 2, 3, &c power of the quantity $\bigcirc$; where $⓪$, or the 0 power, is the beginning of quantity, or arithmetical unit. He also extends this notation to roots or fractional exponents, and even to radical ones.

Thus $\bigcirc\!\!\!\!\!\!\;\tfrac{1}{2}$, $\bigcirc\!\!\!\!\!\!\;\tfrac{1}{3}$, $\bigcirc\!\!\!\!\!\!\;\tfrac{1}{4}$, &c, are the square root, cube root, 4th root, &c;

and $\bigcirc\!\!\!\!\!\!\;\tfrac{2}{3}$ is the cube root of the square;

and $\bigcirc\!\!\!\!\!\!\;\tfrac{3}{2}$ is the square root of the cube. And so of others.

The first three powers, $①$, $②$, $③$, he also calls *coste* (side), *quarre* (square), *cube* (cube); and the first of them,

①, the prime quantity, which he observes is also *metaphorically* called the racine or root, (the mark of which is also $\surd$), because it represents the root or origin from whence all other quantities spring or arise, called the *potences* or powers of it. He condemns the terms sursolids, and numbers absurd, irrational, irregular, inexplicable, or surd, and shows that all numbers are denoted the same way, and are all equally proper expressions of some length or magnitude, or some power of the same root. He also rejects all the compound expressions of square-squared, cube-squared, cube-cubed, &c, and observes that it is best to name them all by their exponents, as the 1st, 2d, 3d, 4th, 5th, 6th, &c power or quantity in the series. And on his extension of the new notation he justly observes, that what was before obscure, laborious, and tiresome, will by these marks be clear, easy, and pleasant. He also makes the notation of algebraic quantities more general in their coefficients, including in them not only integers, as 3①, but also fractions and radicals, as $\frac{3}{4}$②, and $\surd 2$③, &c. He has various other peculiarities in his notations; all showing an original and inventive mind. A quantity of several terms, *he* calls a multinomial, and also binomial, trinomial, &c, according to the number of the terms. He uses the signs $+$ and $-$, and sometimes : for equality; also $\times$ for division of fractions, or to multiply crosswise thus, $\frac{5}{7} \times \frac{2}{3} : \frac{15}{14}$.

He teaches the generation of powers by means of the annexed table of numbers, which are the coefficients of all the terms, except the first and last. And he makes use of the same numbers also for extracting all roots whatever: both which things had first been done by Stifelius. In extracting the roots of non-quadrate or non-cubic numbers, he has the same approximations as at present, viz, either to continue the extraction indefinitely in decimals, by adding periods of ciphers, or by making a fraction of the remainder in this

2
3 3
4 6 4
5 10 10 5
6 15 20 15 6
&c.

manner, viz, $\sqrt{N} = n + \frac{N - n^2}{2n + 1}$ nearly, and $\sqrt[3]{N} = n + \frac{N - n^3}{3n^2 + 3n + 1}$ nearly; where n is the nearest exact root of N; which is Peletarius's rule, and which differs from Tartalea's rule, as this wants the 1 in the denominator. And in like manner he goes on to the roots of higher powers.

He then treats of equations, and their inventors, which according to him are thus:

Mahomet, son of Moses, an Arabian, invented these - - - - { ① egale à ⓪, its derivatives, ② egale à ①, ⓪.

And some unknown author, the derivatives of this.

Some unknown author invented these { ③ egale à ① ⓪, ③ egale à ② ⓪.

But afterwards he mentions Ferreus, Tartalea, Cardan, &c, as being also concerned in the invention of them.

Lewis Ferrari invented - ④ egale à ③ ② ① ⓪.

He says also that Diophantus once resolves the case ② egale à ① ⓪. In his reduction of equations, which is full and masterly, he always places the highest power on one side alone, equal to all the other terms, set in their order, on the other side, whether they be + or −. And he demonstrates all the rules both arithmetically and geometrically. In cubics, he gives up the irreducible case, as hopeless: but says that Bombelli resolves it by *plus of minus*, and *minus of minus*:

thus, if 1 ③ $= 30$ ① $+ 36$, then 1 ① $= \sqrt[3]{}(18 + \text{of} - 26) + \sqrt[3]{}(18 - \text{of} - 26)$, that is, 1 ① $= \sqrt[3]{}(18 + 26\sqrt{-1}) + \sqrt[3]{}(18 - 26\sqrt{-1})$. He resolves biquadratics by means of cubics and quadratics. In quadratics, he takes both the two roots, but looks for no more than two in cubics or biquadratics. He gives also a general method of approaching indefinitely near, in decimals, to the root of any equation whatever; but it is very laborious, being little more than trying all numbers, one after another, finding thus the 1st figure, then the 2d, then the 3d, &c, among these ten characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. And finally he

applies the rules, in the resolution of a great many practical questions.

Though a general air of originality and improvement runs through the whole of Stevinus's work, yet his more remarkable or peculiar inventions, may be reduced to these few following: viz,

1st. He invented not only a new character for the unknown quantity, but greatly improved the notation of powers, by numeral indices, first given by Stifelius as to integral exponents; which Stevinus extended to fractional and all other sorts of exponents, thereby denoting all sorts of roots the same way as powers, by numeral exponents. A circumstance hitherto thought to be of much later invention.

2d. He improved and extended the use and notation of coefficients, including in them fractions and radicals, and all sorts of numbers in general.

3d. A quantity of several terms, he called generally a multinomial; and he denoted all nomials whatever by particular names expressing the number of their terms, binomial, trinomial, quadrinomial, &c.

4th. A numeral solution of all equations whatever, by one general method.

Besides which, he hints at some unknown author as the first inventor of the rules for cubic equations; by whom may perhaps be intended the author of the Arabic manuscript treatise on cubic equations, given to the library at Leyden by the celebrated Warner.

VIETA.

Most of Vieta's algebraical works were written about or before the year 1600, but some of them were not published till after his death, which happened in the year 1603, in the 63d year of his age. And his whole mathematical works were collected together by Francis Schooten, and elegantly printed in a folio volume in 1646. Of these, the algebraical parts are as follow:

1. Isagoge in Artem Analyticam.
2. Ad Logisticen Speciosam Notæ priores.
3. Zeteticorum libri quinque.
4. De Æquationum Recognitione, & Emendatione.
5. De Numerosâ Potestatum ad Exegesin Resolutione.

Of all these I shall give a particular account, especially in such parts as contain any discoveries, as we here meet with more improvements and inventions on the nature of equations, than in almost any former author. And first of the *Isagoge*, or Introduction to the Analytic Art. In this short introduction Vieta lays down certain præcognita in this art; as definitions, axioms, notations, common precepts or operations of addition, subtraction, multiplication, and division, with rules for questions, &c. From which we find, 1st, That the names of his powers are latus, quadratum, cubus, quadrato-quadratum, quadrato-cubus, cubo-cubus, &c; in which he follows the method of Diophantus, and not that derived from the Arabians. 2d, That he calls powers pure or adfected, and first here uses the terms coefficient, affirmative, negative, specious logistics or calculations, homogeneum comparationis, or the absolute known term of an equation, homogeneum adfectionis, or the 2d or other term which makes the equation adfected, &c. 3d, That he uses the capital letters to denote the known as well as unknown quantities, to render his rules and calculations general; namely, the vowels A, E, I, O, U, Y for the unknown quantities, and the consonants B, C, D, &c, for the known ones. 4th, That he uses the sign + between two terms for addition; − for subtraction, placing the greater before the less; and when it is not known which term is the greater, he places = between them for the difference, as we now use $\backsim$; thus A = B is the same as A $\backsim$ B; that he expresses division by placing the terms like a fraction, as at present; though he was not first in this. But that he uses no characters for multiplication or equality, but writes the words themselves, as well as the names of all the powers, as he uses no exponents, which causes much trouble and

prolixity in the progress of his work; and the numeral coefficients set after the literal quantities, have a disagreeable effect.

II. *Ad Logisticen Speciosam, Notæ Priores.* These consist of various theorems concerning sums, differences, products, powers, proportionals, &c, with the genesis of powers from binomial and residual roots, and certain properties of rational right-angled triangles.

III. *Zeteticorum libri quinque.* The zetetics or questions, in these 5 books, are chiefly from Diophantus, but resolved more generally by literal arithmetic. And in these questions are also investigated rules for the resolution of quadratic and cubic equations. In these also Vieta first uses a line drawn over compound quantities, as a vinculum.

IV. *De Æquationum Recognitione, & Emendatione.* These two books, which contain Vieta's chief improvements in Algebra, were not published till the year 1615, by Alexander Anderson, a learned and ingenious Scotchman, with various corrections and additions. The 1st of these two books consists of 20 chapters. In the first six chapters, rules are drawn from the zetetics for the resolution of quadratic and cubic equations. These rules are by means of certain quantities in continued proportion, but in the solution they come to the same thing as Cardan's rules. In the cubics, Vieta sometimes changes the negative roots into affirmative, as Cardan had done, but he finds only the affirmative roots. And he here refers the irreducible case to angular sections for a solution, a method which had been mentioned by Bombelli.

Chap. 7 treats of the general method of transforming equations, which is done either by changing the root in various ways, namely by substituting another instead of it which is either increased or diminished, or multiplied or divided, by some known number, or raised or depressed in some known proportion; or by retaining the same root, and equally multiplying all the terms. Which sorts of transformation, it is evident, are intended to make the

equation become simpler, or more convenient for solution. And all or most of these reductions and transformations were also practised by Cardan.

Chap. 8 shows what purposes are answered by the foregoing transformations; such as taking away some of the terms out of an equation, and particularly the 2d term, which is done by increasing or diminishing the root by the coefficient of the 2d term, divided by the index of the first: by which means also the affected quadratic is reduced to a simple one. And various other effects are produced.

Chap. 9 shows how to deduce compound quadratic equations from pure ones, which is done by increasing or diminishing the root by a given quantity, being one application of the foregoing reductions.

Chap. 10, the reduction of cubic equations affected with the 1st power, to such as are affected with the 2d power; by the same means.

In *chap.* 11, by the same means also, the 2d term is restored to such cubic equations as want it.

In *chap.* 12, quadratic and cubic equations are raised to higher degrees, by substituting for the root, the square or cube of another root divided by a given quantity.

In *chap.* 13, affected biquadratic equations are deduced from affected quadratics in this manner, when expressed in the modern notation: If $A^2 + BA = Z$,

then shall $A^4 + \overline{B^3 + 2BZ} \,.\, A = Z^2 + B^2Z$.

For since $A^2 + BA = Z$, therefore $A^2 = Z - BA$,

and its square is $A^4 = Z^2 - 2BAZ + B^2A^2$:

but $B^2A^2 = B^2Z - B^3A$,

therefore $A^4 = Z^2 - 2BAZ + B^2Z - B^3A$,

or $A^4 + \overline{B^3 + 2BZ} \,.\, A = Z^2 + B^2Z$.

And in like manner for the biquadratic affected with its other terms. And in a similar manner also, in chap. 14, affected cubic equations are deduced from the affected quadratics.

In *chap.* 15 it is shown, that the quadratic $BA - A^2 = Z$

has two values of the root A, or has ambiguous roots, as he calls them; and also that the cubics, biquadratics, &c, which are raised or deduced from that quadratic, have also double roots.

Having, in the foregoing chapters, shown how the coefficients of equations of the 3d and 4th degree are formed from those of the 2d degree, of the same root; and that certain quadratics, and others raised from them, have double roots; then in the 16th chap. Vieta shows what relation those two roots bear to the coefficients of the two lowest terms of an equation consisting of only three terms. Thus,

$$\left.\begin{array}{l}\text{If } BA - A^2 = Z,\\ \text{and } BE - E^2 = Z;\end{array}\right\}\begin{array}{l}\text{then } B = \dfrac{A^2 - E^2}{A - E} = A + E,\\ \text{and } Z = \dfrac{A^2E - AE^2}{A - E} = AE.\end{array}$$

$$\left.\begin{array}{l}\text{If } BA - A^3 = Z,\\ \text{and } BE - E^3 = Z;\end{array}\right\}\begin{array}{l}\text{then } B = \dfrac{A^3 - E^3}{A - E} = A^2 + AE + E^2,\\ \text{and } Z = \dfrac{A^3E - AE^3}{A - E} = A^2E + AE^3.\end{array}$$

$$\left.\begin{array}{l}\text{If } A^m + BA^n = Z,\\ \text{and } E^m - BE^n = Z;\end{array}\right\}\text{then } B = \dfrac{E^m - A^m}{E^n + A^n}, \text{ and } Z = \dfrac{A^mE^n + A^nE^m}{E^n + A^n}.$$

And so on for the same terms with their signs variously changed.

Chap. 17 contains several theorems concerning quantities in continued geometrical progression. Which are preparatory to what follows, concerning the double roots of equations, the nature of which he expounds by means of such properties of proportional quantities.

Chap. 18, *Æquationum ancipitum constitutiva;* treating of the nature of the double roots of equations. Thus, if a, b, c, d, &c, be quantities in continual progression; then, 1st, of equations affected with the first power,

If $BA - A^2 = Z$; then $B = a + b$, $Z = ab$, and $A = a$ or b.

If $BA - A^3 = Z$; then $B = a^2 + b^2 + c^2$, $Z = a(b^2 + c^2)$, and $A = a$ or c.

And in general, if $BA - A^{n+1}$; then $B = a^n + b^n + c^n \cdots k^n$, $Z = a(b^n + c^n + d^n \cdots k^n)$, and $A = a$ or

k, the first or last term. Where the number of terms a^n, b^n, &c, in B, is $n+1$, and the number of terms in Z is n.

2d, For equations containing only the highest two powers.
If $BA^2 - A^3 = Z$; then $B = a + b + c$, $Z = a(b+c)^2$, or $= c(a+b)^2$, and $A = a + b$, or $b + c$.
If $BA^3 - A^4 = Z$; then $B = a + b + c + d$, $Z = a(b+c+d)^3$ or $= d(a+b+c)^3$, and $A = a + b + c$ or $= b + c + d$.
And, in general, if $BA^n - A^{n+1} = Z$;
then $B = a + b + c \dots i + k$,
$Z = a(b + c \dots k)$ or $= k(a + b + c \dots i)$,
and $A = a + b + c \dots i$ or $= b + c + d \dots k$, the sum of all except the last, or sum of all except the first; where the number of terms in B, is $n+1$, and the number of terms in Z, is n.

3d. Of equations affected by the intermediate powers.
If $BA^2 - A^4 = Z$; then $B = a^2 + b^2$, $Z = a^2 b^2$, & $A^2 = a^2$ or b^2.
If $BA^3 - A^6 = Z$; then $B = a^3 + b^3$, $Z = a^3 b^3$, & $A^3 = a^3$ or b^3.
If $BA^4 - A^6 = Z$; then $B = a^2 + b^2 + c^2$, $Z = a(b^2 + c^2)$, and $A^2 = a + b$ or $b + c$.

4th. Of the remaining cases.
If $BA^2 - A^5 = Z$;
then $B = (a+b)^3 + (b+c)^3 + a(b+c)^2$ or $+ c(a+b)^2$,
and $Z = \overline{B - (a+b)^3} \cdot (a+b)^2$
or $= \overline{(b+c)^3 + c(a+b)^2} \cdot (a+b)^2$,
and $A = a + b$ or $b + c$.
If $BA^3 - A^5 = Z$;
then $B = (a+b+c)^2 + (b+c+d)^2 - c(a+b+c)$ or $- b(b+c+d)$,
and $Z = \overline{B - (b+c+d)^2} \cdot (b+c+d)^3$
or $= \overline{B - (a+b+c)^2} \cdot (a+b+c)^3$;
and $A = a + b + c$ or $b + c + d$.

Chap. 19. *Æqualitatum contradicentium constitutiva.* Of the relation of equations of like terms, but the sign of one term different; containing these 5 theorems, viz.

1. If $A^2 + BA = Z$, and $E^2 - BE = Z$; $\Big\}$ then $B = b - a$, $Z = ab$; and $A = a$, $E = b$.

$$\left.\begin{array}{l} \text{2. If } A^4 + BA = Z, \\ \text{and } E^4 - BE = Z; \end{array}\right\}\begin{array}{l} \text{then } B = (b+d)^3 - (a+c)^3, \\ \quad Z = a(d^3 - b^3) \text{ or } = d(c^3 - a^3); \\ \text{and } A = a,\ E = d. \end{array}$$

$$\left.\begin{array}{l} \text{3. If } A^6 + BA = Z, \\ \text{and } E^6 - BE = Z; \end{array}\right\}\begin{array}{l} \text{then } B = b^5 + d^5 + f^5 - a^5 - c^5 - e^5, \\ \quad Z = a(b^5 + d^5 + f^5 - c^5 - e^5), \\ \quad \text{or } = f(a^5 + c^5 + e^5 - b^5 - d^5); \\ \text{and } A = a,\ E = f. \end{array}$$

$$\left.\begin{array}{l} \text{4. If } A^4 + BA^3 = Z, \\ \text{and } E^4 - BE^3 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = b + d - a - c, \\ \quad Z = a(d-b)^3, \text{ or } = d(c-a)^3; \\ \text{and } A = c - a,\ E = d - b. \end{array}$$

$$\left.\begin{array}{l} \text{5. If } A^6 + BA^5 = Z, \\ \text{and } E^6 - BE^5 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = b + d + f - a - c - e; \\ \quad Z = a(b + d + f - c - e)^5, \\ \quad \text{or } = f(a + c + e - b - d)^5; \\ \text{and } A = a + c + e - b - d, \\ \quad E = b + d + f - c - e. \end{array}$$

Chap. 20. *Æqualitatum inversarum constitutiva.* Containing these six theorems, viz,

$$\left.\begin{array}{l} \text{1. If } BA - A^3 = Z, \\ \text{and } E^3 - BE = Z; \end{array}\right\}\begin{array}{l} \text{then } B = c^2 - a^2, \\ \quad Z = a(c^2 - b^2), \text{ or } = c(b^2 - a^2); \\ \text{and } A = a,\ E = c. \end{array}$$

$$\left.\begin{array}{l} \text{2. If } BA - A^5 = Z, \\ \text{and } E^5 - BE = Z; \end{array}\right\}\begin{array}{l} \text{then } B = a^4 + c^4 + e^4 - b^4 - d^4, \\ \quad Z = a(c^4 + e^4 - b^4 - d^4), \\ \quad \text{or } = e(b^4 + d^4 - a^4 - c^4); \\ \text{and } A = a,\ E = e. \end{array}$$

$$\left.\begin{array}{l} \text{3. If } BA^2 + A^3 = Z, \\ \text{and } BE^2 - E^3 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = c - a,\ Z = a(c-b)^2, \\ \quad \text{or } = c(b-a)^2; \\ \text{and } A = b - a,\ E = c - b. \end{array}$$

$$\left.\begin{array}{l} \text{4. If } BA^4 + A^5 = Z, \\ \text{and } BE^4 - E^5 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = a + c + e - b - d, \\ \quad Z = a(c + e - b - d)^4, \\ \quad \text{or } = e(b + d - a - c)^4; \\ \text{and } A = b + d - a - c, \\ \quad E = c + e - b - d. \end{array}$$

$$\left.\begin{array}{l} \text{5. If } BA^3 + A^5 = Z, \\ \text{and } BE^3 - E^5 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = (a + f)\,.\,(d - a), \\ \quad Z = (d - a)^3 B + (d - a)^5, \\ \quad \text{or } = (e - b)^3 B - (e - b)^5; \\ \text{and } A = d - a,\ E = e - b. \end{array}$$

$$\left.\begin{array}{l} \text{6. If } BA^2 + A^5 = Z, \\ \text{and } BE^2 - E^5 = Z; \end{array}\right\}\begin{array}{l} \text{then } B = (a + f)\,.\,(c - a)^2, \\ \quad Z = \overline{B + (c - a)^3}\,.\,(c - a)^2, \\ \quad \text{or } = \overline{B - (d - b)^3}\,.\,(d - b)^2, \\ \text{and } A = c - a,\ E = d - b. \end{array}$$

Chap. 21. *Alia rursus æqualitatum inversarum constitutiva.* In these two theorems:

1. If $B^2A - A^3 = Z$, and $E^3 - B^2A = Z$; then $B = a^2 + b^2 + c^2$, $Z = (a + c)b^2$ or $= a(b^2 + c^2)$, or $= c(a^2 + b^2)$; and $A = a$ or c, $E = a + c$.

2. If $BA^2 - A^3 = Z$, and $BE^2 + E^3 = Z$; then $B = a + b + c$, $Z = (a + b + c)b^2$ or $= a(b + c)^2$ or $= c(a + b)^2$; and $A = a + b$ or $b + c$, $E = b$.

Next follows the 2d of the pieces published by Alexander Anderson, namely,

De Emendatione Æquationum, in 14 chapters.

Chap. 1. Of preparing equations for their resolution in numbers, by taking away the 2d term; by which affected quadratics are reduced to pure ones, and cubic equations affected with the 2d term are reduced to such as are affected with the 3d only. Several examples of both sorts of equations are given. He here too remarks upon the method of taking away any other term out of an equation, when the highest power is combined with that other term only; and this Vieta effects by means of the coefficients, or, as he calls them, the unciæ of the power of a binomial. All which was also performed by Cardan for the same purpose.

Chap. 2. *De transmutatione* Πρῶτον—ἔσχατον, *quæ remedium est adversus vitium negationis.* Concerning the transformations by changing the given root A for another root E, which is equal to the homogeneum comparationis divided by the first root A; by which means negative terms are changed to affirmative, and radicals are taken out of the equation when they are contained in the homogeneum comparationis.

Chap. 3. *De Anastrophe,* showing the relation between the roots of correlate equations; whence, having given the root of the one equation, that of the other becomes known; and it consists of these following 8 theorems, mostly deduced from the last 4 chapters of the foregoing *recognitio æquationum.*

$$1.\ \left.\begin{array}{l}\text{If } BA - A^3 = Z,\\ \text{and } E^3 - BE = Z;\end{array}\right\}\text{then } EA - A^2 = E^2 - B.$$

$$2.\ \left.\begin{array}{l}\text{If } BA^2 - A^3 = Z,\\ \text{and } BE^2 + E^3 = Z;\end{array}\right\}\text{then } (E + B)A - A^2 = BD + D^2.$$

$$3.\ \left.\begin{array}{l}\text{If } BA - A^5 = Z,\\ \text{and } E^3 - BE = Z;\end{array}\right\}\text{then } E^3A - E^2A^2 + EA^3 - A^4 = E^4 - B.$$

$$4.\ \left.\begin{array}{l}\text{If } BA^4 - A^5 = Z,\\ \text{and } BE^4 + E^5 = Z;\end{array}\right\}\begin{array}{l}\text{then } (BE^2 + E^3)A - (BE + E^2)A^2\\ \quad + (B + E)A^3 - A^4 = BE^3 + E^4.\end{array}$$

$$5.\ \left.\begin{array}{l}\text{If } A^3 - BA = Z,\\ \text{and } BE - E^3 = Z;\end{array}\right\}\text{then } A^2 - EA = B - E^2.$$

$$6.\ \left.\begin{array}{l}\text{If } BA^2 + A^3 = Z,\\ \text{and } BE^2 - E^3 = Z;\end{array}\right\}\text{then } A^2 + (B - E)A = BE - E^2.$$

$$7.\ \left.\begin{array}{l}\text{If } BA - A^3 = Z,\\ \text{and } BE - E^3 = Z;\end{array}\right\}\text{then } A^2 + EA = B - E^2.$$

$$8.\ \left.\begin{array}{l}\text{If } BA^2 - A^3 = Z,\\ \text{and } BE^2 - E^3 = Z;\end{array}\right\}\text{then } A^2 + (E - B)A = BE - E^2.$$

Chap. 4. *De Isomæria, adversus vitium fractionis.* To take away fractions out of an equation. Thus,

if $A^3 + \frac{B}{D}A = Z$. Put $A = \frac{E}{D}$; then $E^3 + BDE = ZD^3$.[1]

Chap. 5. *De Symmetrica Climactismo adversus vitium asymmetriæ.* To take away radicals or surds out of equations, by squaring &c the other side of the equation.

Chap. 6. To reduce biquadratic equations by means of cubics and quadratics, by methods which are small variations from those of Ferrari and Cardan.

Chap. 7. The resolution of cubic equations by rules which are the same with Cardan's.

Chap. 8. *De Canonica æquationum transmutatione, ut coefficientes subgraduales sint quæ præscribuntur.* To transmute the equation so that the coefficient of the lower term, or power, may be any given number, he changes the root in the given proportion, thus: Let A be the root of the equation given, E that of the transmuted equation, B the given coefficient, and X the required one; then take $A = \frac{BE}{X}$, which substitute in the given equation, and it is done.—He commonly changes it so, that X may be 1; which he does, that the numeral root of the equation may be the easier found; and

this he here performs by trials, by taking the nearest root of the highest power alone; and if that does not turn out to be the root of the whole equation, he concludes that it has no rational root.

Chap. 9. To reduce certain peculiar forms of cubics to quadratics, or to simpler forms, much the same as Cardan had done. Thus,

1. If $A^3 - 2B^2A = B^3$; then is $A^2 - BA = B^2$.
2. If $2B^2A - A^3 = B^3$; then is $A^2 + BA = B^2$.
3. If $A^3 - 3B^2A = 2B^3$; then is $A = 2B$.
4. If $3B^2A - A^3 = 2B^3$; then is $A = B$.
5. If $A^3 - BA^2 + DA = BD$; then is $A = B$.
6. If $A^3 + BA^2 - D^2A = BD^2$; then is $A = D$.
7. If $BA^2 + D^2A - A^3 = BD^2$; then is $A = B$ or $= D$.
8. If $D^2A + BDA - A^3 = B^3$; then is $(D + B)A - A^2 = B^2$.
9. If $A^3 - DA^2 + BDA = B^3$; then is $(D - B)A - A^2 = B^2$.
10. If $A^3 - 3BA = B\sqrt{2B}$; then is $A = \sqrt{\tfrac{3}{2}B} - \sqrt{\tfrac{1}{2}B}$.
11. If $3BA - A^3 = B\sqrt{2B}$; then is $A = \sqrt{\tfrac{3}{2}B} - \sqrt{\tfrac{1}{2}B}$.

Chap. 10. *Similium reductionum continuatio.* Being some more similar theorems, when the equation is affected with all the powers of the unknown quantity A.

Chap. 11, 12, 13 relate also to certain peculiar forms of equations, in which the root is one of the terms of a certain series of continued proportionals.

Chap. 14, which is the last in this tract, contains, in four theorems, the general relation between the roots of an equation and the coefficients of its terms, when all its roots are positive. Namely,

1. If $\overline{B + D} . A - A^2 = BD$; then is $A = B$ or D.
2. If $A^3 \overline{- B - D - G} . A^2 + \overline{BD + BG + DG} . A = BDG$; then is $A = B$ or D or G.
3. If $\overline{BDG + BDH + BGH + DGH} . A \overline{- BD - BG - BH - DG - DH - GH} . A^2 + \overline{B + D + G + H} . A^3 - A^4 = BDGH$; then is $A = B$ or D or G or H.
4. If $A^5 \overline{- B - D - G - H - K} . A^4 + \overline{BD + BG + BH + BK + DG + DH + DK + GH + GK + HK} . A^3 \overline{- BDG - BDH -}$

$\overline{\text{BDK}-\text{BGH}-\text{BGK}-\text{BHK}-\text{DGH}-\text{DGK}-\text{DHK}-\text{GHK}}\,.\,\text{A}^2$
$\overline{+\text{BDGH}+\text{BDGK}+\text{BDHK}+\text{BGHK}+\text{DGHK}}\,.\,\text{A}$
$=\text{BDGHK}$; then is $\text{A}=\text{B}$ or D or G or H or K.

And from these last 4 theorems it appears, that Vieta was acquainted with the composition of these equations, that is, when all their roots are positive, for he never adverts to negative roots; and from other parts of the work it appears, that he was not aware that the same properties will obtain in all sorts of roots whatever. But it is not certain in what manner he obtained these theorems, as he has not given any account of the investigations, though that was usually his way on other occasions; but he here contents himself with barely announcing the theorems as above, and for this strange reason, that he might at length bring his work to a conclusion.

To this piece is added, by Alexander Anderson, an Appendix, containing the construction of cubic equations by the trisection of an angle, and a demonstration of the property referred to by Vieta for this purpose.

De Numerosa Protestàtum Purarum Resolutione. Vieta here gives some examples of extracting the roots of pure powers, in the way that had been long before practised, by pointing the number into periods of figures according to the index of the root to be extracted, and then proceeding from one period to another, in the usual way.

De Numerosa Potestatum adfectarum Resolutione. And here, in close imitation of the above method for the roots of pure powers, Vieta extracts those of adfected ones; or finding the roots of affected equations, placing always the homogeneum comparationis, or absolute term, on one side, and all the terms affected with the unknown quantity, and their proper signs, on the other side. The method is very laborious, and is but little more than what was before done by Stevinus on this subject, depending not a little upon trials. The examples he uses are such as have either one or two roots, and indeed such as are affected commonly with only two powers of the unknown quantity, and which therefore admit only of these two varieties as to the

number of roots, namely according as the higher of the two powers is affirmative or negative, the homogeneum comparationis, on the other side of the equation, being always affirmative ; and he remarks this general rule, if the higher power be negative, the equation has two roots ; otherwise, only one; that is, affirmative roots ; for as to negative and imaginary ones, Vieta knew nothing about them, or at least he takes no notice of them. By the foregoing extraction, Vieta finds both the greater and less root of the two that are contained in the equation, and either of them that he pleases; having first, for this purpose, laid down some observations concerning the limits within which the two roots are contained. Also, having found one of the roots, he shows how the other root may be found by means of another equation, which is a degree lower than the given one; though not by depressing the given equation, by dividing it as is now done ; but from the nature of proportionals, and the theorems relating to equations, as given in the former tracts, he finds the terms of another equation, different from that last mentioned, from the root &c of which, the 2d root of the original equation may be obtained.

In the course of this work, Vieta makes also some observations on equations that are ambiguous, or have three roots ; namely, that the equation 1C − 6Q + 11N = 6, or as we write it $x^3 - 6x^2 + 11x = 6$, is ambiguous, when the 2d term is negative, and the 3d term affirmative, and when $\frac{1}{3}$ of the square of 6, the coefficient of the 2d term, exceeds 11, the coefficient of the 3d term, and has then three roots. Or in general, if $x^3 - ax^2 + bx = c$, and $\frac{1}{3}a^2 > b$, the equation is ambiguous, and has three roots. He shows also, from the relation of the coefficients, how to find whether the roots are in arithmetical progression or not, and how far the middle root differs from the extremes, by means of a cubic equation of this form $x^3 - bx = c$. In all or most of which remarks he was preceded by Cardan.—Vieta also remarks that the case $x^3 - 9x^2 + 24x = 20$, has three roots by the same rule, viz, 2, 2, 5, but that two of them are equal.

And further, that when $\frac{1}{3}a^2$ is $= b$, then all the three roots are equal, as in the case $x^3 - 6x^2 + 12x = 8$, the three roots of which are 2, 2, 2. But when $\frac{1}{3}a^2$ is less than b, the case is not ambiguous, having but one root. And when $ab = c$, then $a = x$ is one root itself.

Many curious notes are added at the end, with remarks on the method of finding the approximate roots, when they are not rational, which is done in two ways, in imitation of the same thing in the extraction of pure powers, viz, the one by forming a fraction of the remainder after all the figures of the homogeneum comparationis are exhausted ; the other by increasing the root of the equation in a 10 fold, or 100 fold, &c, proportion, and then dividing the root which results by 10, or 100, &c : and this is a decimal approximation. And Vieta observes that the roots will be increased 10 or 100 fold, &c, by adding the corresponding number of ciphers to the coefficient of the 2d term, double that number to the 3d, triple the same number to the 4th, and so on. So if the equation were

1 C + 4Q + 6N = 8,

then 1 C + 40Q + 600N = 8000 will have its root 10 fold,
then 1 C + 400Q + 60000 N = 8000000 will have it 100 fold.

Besides the foregoing algebraical works, Vieta gave various constructions of equations by means of circles and right lines, and angular sections, which may be considered as an algebraical tract, or a method of exhibiting the roots of certain equations having all their roots affirmative, and by means of which he resolved the celebrated equation of 45 powers, proposed to all the world by Adrianus Romanus.

Having now delivered a particular analysis of Vieta's algebraical writings, it will be proper, as with other authors, to collect into one view the particulars of his more remarkable peculiarities, inventions, and improvements.

And first it may be observed, that his writings show great originality of genius and invention, and that he made alterations and improvements in most parts of algebra ; though in other parts and respects his method is inferior to some of his

predecessors; as, for instance, where he neglects to avail himself of the negative roots of Cardan; the numeral exponents of Stifelius, instead of which he uses the names of the powers themselves; or the fractional exponents of Stevinus; or the commodious way of prefixing the coefficient before the quantity or factor; and such like circumstances; the want of which gives his Algebra the appearance of an age much earlier than its own. But his real inventions of things before not known, may be reduced to the following particulars.

1st. Vieta introduced the general use of the letters of the alphabet, to denote indefinite given quantities; which had only been done on some particular occasions before his time. But the general use of letters, for the unknown quantities, was before pretty common with Stifelius and his successors. Vieta uses the vowels A, E, I, O, U, Y for the unknown quantities, and the consonants B, C, D, &c, for known ones.

2d. He invented, and introduced many expressions or terms, several of which are in use to this day: such as coefficient, affirmative and negative, pure and adfected or affected, unciæ, homogeneum adfectionis, homogeneum comparationis, the line or vinculum over compound quantities, thus $\overline{A + B}$. And his method of setting down his equations, is to place the homogeneum comparationis, or absolute known term, on the right-hand side alone, and on the other side all the terms which contain the unknown quantity, with their proper signs.

3d. In most of the rules and reductions for cubic and other equations, he made some improvements, and variations in the modes.

4th. He showed how to change the root of an equation in a given proportion.

5. He derived or raised the cubic and biquadratic, &c equations, from quadratics; not by composition in Harriot's way, but by squaring and otherwise multiplying certain parts of the quadratic. And as some quadratic equations have two roots, therefore the cubics and others raised from

them, have also the same two roots, and no more. And hence he comes to know what relation these two roots bear to the coefficients of the two lowest terms of cubic and other equations, when they have only 3 terms, namely, by comparing them with similar equations so raised from quadratics. And, on the contrary, what the roots are, in terms of such coefficients.

6. He made some observations on the limits of the two roots of certain equations.

7. He stated the general relation between the roots of certain equations and the coefficients of its terms, when the terms are alternately plus and minus, and none of them are wanting, or the roots all positive.

8. He extracted the roots of affected equations, by a method of approximation similar to that for pure powers.

9. He gave the construction of certain equations, and exhibited their roots by means of angular sections; before adverted to by Bombelli.

OF ALBERT GIRARD.

Albert Girard was an ingenious Dutch or Flemish mathematician, who died about the year 1633. He published an edition of Stevinus's Arithmetic in 1625, augmented with many notes; and the year after his death was published by his widow, an edition of the whole works of Stevinus, in the same manner, which Girard had left ready for the press. But the work which entitles him to a particular notice in this history, is his "*Invention Nouvelle en l'Algebre, tant pour la solution des equations, que pour recognoistre le nombre des solutions qu'elles reçoivent, avec plusieurs choses qui sont necessaires a la perfection de ceste divine science;*" which was printed at Amsterdam 1629, in small quarto, in 63 pages, viz, 49 pages on Arithmetic and Algebra, and the rest on the measure of the superficies of spherical triangles and polygons, by him then lately discovered.

In this work, Girard first premises a short tract on Arith-

metic ; in the notation of which he has something peculiar, viz, dividing the numbers into the ranks of millions, billions, trillions, &c.

He next delivers the common rules of Algebra, not only in integers and fractions, but radicals also ; with the notation of the quantities and signs. In this part he uses sometimes the letters A, B, C, &c, after the manner of Vieta, but more commonly the characters of Stevinus, viz, $\textcircled{0}$, $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$, &c, for the powers of the unknown quantity, with their roots $\textcircled{\tfrac{3}{2}}$, $\textcircled{\tfrac{1}{2}}$, $\textcircled{\tfrac{1}{4}}$, $\textcircled{\tfrac{2}{3}}$, $\textcircled{\tfrac{5}{4}}$, &c, used by Stevinus ; and sometimes the more usual marks of the roots, as $\surd$ or $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, &c ; prefixing the coefficients, as $6\textcircled{2}$, or $3\sqrt[5]{3^2}$, or $2\textcircled{\tfrac{1}{2}}$. In the signs he follows his predecessors so far as to have + for plus, − or ÷ for minus, = for general or indefinite difference, A + B for the sum, A − B or A = B for the difference, AB the product, and $\frac{A}{B}$ for the quotient of A and B. He uses the parentheses ( ) for the vinculum or bond of compound quantities, as is now commonly practised ; as A(AB + Bq), or $\sqrt[3]{}$(A cub. − 3AqB) ; and he introduces the new characters ff for *greater than*, and § for *less than;* but he uses no character for equality, only the word itself.

Girard gives a new rule for extracting the cube root of binomials, which however is in a good measure tentative, and which he explains thus: To extract the cube root of $72 + \sqrt{5120}$.

$$\text{The squares of the terms } \begin{cases} 5184 \\ 5120 \end{cases}$$

their difference 64, and its cube root 4. Which shows that the difference between the squares of the terms required is 4 ; and the rational part 72 being the greater, the greater term of the root will be rational also ; and further, that the greater terms of the power and root are commensurable, as also the two less terms. Then having made a table as in the margin, where the square of the rational term always exceeds that of the other, by the number 4

$2 + \sqrt{0}$
$3 + \sqrt{5}$
$4 + \sqrt{12}$
$5 + \sqrt{21}$

above mentioned, one of these binomials must be the cubic root sought, if the given quantity have such a root; and it must be one of these four forms; for it is known to be carried far enough by observing that the cube root of 72 is less than 5, and the cube root of 5120 less than 21; indeed, this being the case, the last binomial is excluded, as evidently too great; and the first is excluded because one of its terms is 0; therefore the root must be either $3 + \sqrt{5}$ or $4 + \sqrt{12}$. And to know whether of these two it must be, try which of them has its two terms exact divisors of the corresponding terms of the given quantity; then it is found that 3 and 4 are both divisors of 72, but that only 5, and not 12, is a divisor of 5120; therefore $3 + \sqrt{5}$ is the root sought, which upon trial is found to answer. It is remarkable here, that Girard uses $4 + \sqrt{20}$ instead of $4 + \sqrt{12}$, and $5 + \sqrt{29}$ instead of $5 + \sqrt{20}$, contrary to his own rule.

Girard then gives distinct and plain rules for bringing questions to equations, and for the reduction of those equations to their simplest form, for solution, by the usual modes, and also by the way called by Vieta *Isomeria*, multiplying the terms of the equation by the terms of a geometrical progression, by which means the roots are altered in the proportion of 1 to the ratio of the progression. He then treats of the methods of finding the roots of the several sorts of equations, quadratic, cubic, &c; and adds remarks on the proper number of conditions or equations for limiting questions. The quadratics are resolved by completing the square, and both the positive and negative roots are taken; and he observes that sometimes the equation is impossible, as ② equ. 6① — 25, whose roots, he adds, are $3 + \sqrt{-16}$ and $3 - \sqrt{-16}$.

The cubic equations he resolves by Cardan's rule, except the irreducible case, which he the first of any resolves by a table of sines; the other cases he also resolves by tables of sines and tangents; and adds geometrical constructions by means of the hyperbola or the trisection of angles.

He next adds a particular mode of resolving all sorts of equations, that have rational roots, on the principle of the roots being divisors of the last or absolute term, as before mentioned by Peletarius; and then gives the method of approximating to other roots that are not rational, much in the same way as Stevinus.

Having found one root of an equation, by any of the former methods, by means of it he depresses the equation one degree lower, then finds another root, and so on till they are all found; for he shows that every algebraic equation, admits of as many solutions or roots, as there are units in the index of the highest power, which roots may be either positive or negative, or imaginary, or, as he calls them, greater than nothing, or less than nothing, or involved; so the roots of the equation 1③ equ. 7① − 6, are 2, 1, and − 3; and the roots of the equation 1④ equ. 4① − 3 are

$$\begin{array}{c} 1, \\ 1, \\ -1 + \sqrt{-2}, \\ -1 - \sqrt{-2}. \end{array}$$

In depressing an equation to lower degrees, he does not use the method of resolution of Harriot, but that which is derived from the general relation of the roots and coefficients of the terms, which he here fully and universally states, viz, that the coefficient of the 2d term is equal to the sum of all the roots; that of the 3d term equal to the sum of all the products of the roots, taken two by two; that of the 4th term, the sum of the products, taken three by three; and so on, to the last or absolute term, which is the continual product of all the roots; a property which was before stated by Vieta, as to the equations that have all their roots positive; and here extended by Girard to all sorts of roots whatever: but how either Vieta or he came by this property, no where appears that I know of. From this general property, among other deductions, Girard shows how to find the sums of the powers of the roots of an equation; thus, let A, B, C, D, &c, be the 1st, 2d, 3d, 4th, &c, coefficient,

after the first term, or the sums of the products taken one by one, two by two, three by three, &c; then, in all sorts of equations,

$$\left.\begin{array}{l} A \\ Aq - 2B \\ A\text{ cub.} - 3AB + 3C \\ Aqq - 4AqB + 4AC + 2Bq - 4D \end{array}\right\} \text{will be the sum of the} \left\{\begin{array}{l}\text{roots,}\\ \text{squares,}\\ \text{cubes,}\\ \text{biquadrates.}\end{array}\right.$$

Girard next explains the use of negative roots in Geometry, showing that they represent lines only drawn in a direction contrary to those representing the positive roots; and he remarks that this is a thing hitherto unknown. He then terminates the Algebra by some questions having two or more unknown quantities. And he subjoins to the whole a tract on the mensuration of the surfaces of spherical triangles and polygons, by him lately discovered.

From the foregoing account it appears that,

1st, He was the first person who understood the general doctrine of the formation of the coefficients of the powers, from the sums of their roots, and their products, &c.

2d, He was the first who understood the use of negative roots in the solution of geometrical problems.

3d, He was the first who spoke of the imaginary roots, and understood that every equation might have as many roots, real and imaginary, and no more, as there are units in the index of the highest power. And he was the first who gave the whimsical name of *quantities less than nothing* to the negative.

4th, He was also the first who discovered the rules for forming the powers of the roots of any equation.

OF HARRIOT.

Thomas Harriot, an excellent astronomer, philosopher, and mathematician, was born at Oxford in 1560. After taking the degree of bachelor of arts in 1579, he accompanied the famous Sir Walter Raleigh in an expedition to Virginia, where the first English establishment was made.

Harriot here drew the map of the country, and in 1588 gave a relation of the voyage. Being returned to his country, it appears that he gave himself up entirely to the study of the mathematics, and especially to that of algebra. He remained not long unknown to the Earl of Northumberland, a great encourager of the sciences, who maintained several learned men, such as Rob. Hues, Walter Warner, and Nathaniel Tarporley. This enlightened nobleman entertained Harriot in his house, with a salary of £300 sterling a year, a great sum in those times; and in this situation it was that Harriot finished his days, in the year 1621, at 61 years of age. It appears by Kepler's letters, that he held a correspondence with this astronomer, chiefly on the theory of the rainbow. Harriot's manuscripts, lately discovered in the castle of the Earl of Egremont, inform us of many of his astronomical observations, and particularly of those of the spots in the sun as early as the beginning of December 1610, while the first of those of Galileo were not made before the preceding month. So that Harriot must then either have made himself a telescope, or procured one from Holland. He made in the same year also, observations on Jupiter's satellites, and on the remarkable comets of the years 1607 and 1618.

His Algebra was left behind him unpublished, as well as those other papers, at his death, which happened in the year 1621, as before mentioned, and but 6 years after the first publication of the principal parts of Vieta's Algebra, by Alexander Anderson; so that it is probable that Harriot's Algebra was written long before this time, and indeed that he had never seen these pieces.

Harriot's Algebra was published by his friend Walter Warner, in the year 1631: and it would doubtless be highly grateful to the learned in these sciences, if his other curious algebraical and astronomical works were published, from his original papers in the possession of the Earl of Egremont, to whom they have descended from Henry Percy, the Earl of Northumberland, that noble Mæcenas of his day. The book

is in folio, and intitled *Artis Analyticæ Praxis, ad Æquationes Algebraicas nova, expedita, & generali methodo, resolvendas*; a work in all parts of it showing marks of great genius and originality, and is the first instance of the modern form of Algebra, in which it has ever since appeared. It is prefaced by 18 definitions, which are these: 1st, Logistica Speciosa; 2d, Equation; 3d, Synthesis; 4, Analysis; 5, Composition and Resolution; 6, Forming an Equation; 7, Reduction of an Equation; 8, Verification; 9, Numerosa & Speciosa; 10, Excogitata; 11, Resolution; 12, Roots; 13 and 14, The kinds and generation of equations by multiplication, from binomial roots or factors, called original equations.

$$\text{as}\ \begin{array}{c|l} a+b & = aa + ba \\ \underline{a-c} & \quad - ca - bc, \end{array}$$

$$\text{or}\ \begin{array}{c|l} a+b & = aaa + baa + bca \\ a+c & \quad + caa - bda \\ \underline{a-d} & \quad - daa - cda - bcd; \end{array}$$

when he puts a for the unknown quantity, and the small consonants, b, c, d, &c, for its literal values or roots; 15, The first form of canonical equations, which are derived from the above originals, by transposing the homogeneum, or absolute term,

$$\text{thus}\ \begin{array}{l} aa + ba \\ \quad - ca = + bc, \text{ \&c.} \end{array}$$

16, The secondary canonicals, formed from the primary by expelling the 2d term,

$$\text{thus}\ aa = + bb,$$

$$\text{or}\ \begin{array}{l} aaa - bba \\ \quad - bca \\ \quad - cca = + bbc \\ \qquad\qquad + bcc. \end{array}$$

17. That these are called canonicals, because they are adapted to canons or rules for finding the numeral roots, &c. 18, Reciprocal equations, in which the homogeneum is the product of the coefficients of the other terms, and the first term, or highest power of the root, is equal to the product of the powers in the other terms, as $aaa - caa + bba = + bbc$

After these definitions, the work is divided into two principal parts; 1st, of various generations, reductions, and preparations of equations for their resolution in the 2d part. The former is divided into 6 sections as follows.

Sect. 1. *Logistices Speciosæ*, exemplified in the 4 operations of addition, subtraction, multiplication, and division; as also the reduction of algebraic fractions, and the ordinary reduction of irregular equations to the form proper for the resolution of them, namely, so that all the unknown terms be on one side of the equation, and the known term on the other, the powers in the terms ranged in order, the greatest first, and the first or highest power made positive, and freed from its coefficient;

$$\text{as } aa + ba = cd,$$
$$\text{or } aaa + baa - cda = - c^2d.$$

In this part he explains some unusual characters which he introduces, namely,

$=$ for equality, as $a = b$.
$>$ for majority, as $a > b$,
$<$ for minority, as $a < b$;

but the first had been before introduced by Robert Recorde.

Sect. 2. The generation of original equations from binomial factors or roots, and the deducing of canonicals from the originals. He supposes that every equation has as many roots as dimensions in its highest power; then supposing the values of the unknown letter a in any equation to be $b, c, d, f,$ &c, that is $a = b$, and $a = c$, and $a = d$, &c; by transposition, or equal subtraction, these become $a - b = 0$, and $a - c = 0$, and $a - d = 0$, &c, or the same letters with contrary signs for negative values or roots; then two of these binomial factors, multiplied together, give a quadratic equation, three of them a cubic, four of them a biquadratic, and so on, with all the terms on one side of the equation, and 0 on the other side, since, every binomial factor being $= 0$, the continual product of all of them must also be $= 0$. Thus,

$$\left.\begin{array}{l} a+b \\ a+c \\ a-d \end{array}\right| \begin{array}{l} = aaa + baa + bca \\ \quad\quad + caa - bda \\ \quad\quad - daa - cda - bcd = 0 \end{array}$$

an original equation,

and

$$\begin{array}{l} aaa + baa + bca \\ \quad + caa - bda \\ \quad - daa - cda = + bcd \end{array}$$

its canonical, deduced from it. And these operations are carried through all the cases of the 2d, 3d and 4th powers, as to the varieties of the signs + and −, and the proportions of the roots as to equal and unequal, with the reciprocals, &c. From which are made evident, at one glance of the eye, all the relations and properties between the roots of equations, and the coefficients of the terms.

Sect. 3. *Æquationum canonicarum secundariarum a primariis reductio per gradus alicujus parodici sublationem radice supposititia invariata manente.* Containing a great many examples of preparing equations, by taking away the 2d, 3d, or any other of the intermediate terms, which is done by making the positive coefficients in that term, equal to the negative ones, by which means the whole term vanishes, or becomes equal to nothing.

They are extended as far as equations of the 5th degree; and at the end are collected, and placed in regular order, all the secondary canonicals, so reduced; so that by the uniform law which is visible through them all, the series may be continued to the higher degrees as far as we please.

Sect. 4. *Æquationum canonicarum tam primariarum, quam secundariarum, radicum designatio.* A great many literal equations are here set down, and their roots assigned from the form of the equation, that is, all their positive roots; for their negative roots are not noticed here; and it is every where proved that they cannot have any more positive roots than these, and consequently the rest are negative. That those are roots, he proves by substituting them instead of the unknown letter a in the equation, when they make all the terms on one side come to the same thing as the homogeneum on the other side.

Sect. 5. In qua æquationum communium per canonicarum æquipollentiam, radicum numerus determinatur. On the number of the roots of common equations, that is the positive roots. This Harriot determines by comparing them with the like cases found among his canonical forms, which two equations, having the same number of terms with the same signs, and the relations of the coefficients and homogeneum correspondent, he calls equipollents. And whatever was the number of positive roots used in the composition of the canonical, the same, he infers, is the number in the proposed common equation. It is remarkable that, in all the examples here used, the number of positive roots is just equal to the number of the changes in the signs from + to − and from − to +, which is a circumstance, though not here expressly mentioned, that could not escape the observation, or the eye, of any one, much less of so clear and comprehensive a sight as that of Harriot.

In this section are contained many ingenious disquisitions, concerning the limits and magnitudes of quantities, with several curious lemmas, laid down to demonstrate the propositions by, which lemmas are themselves demonstrated in a pure mathematical way, from the magnitudes themselves, independent of geometrical figures; such as, 1, If a quantity be divided into any two unequal parts, the square of half the quantity will be greater than the product of the two unequal parts. 2, In three continued proportionals, the sum of the extremes is greater than double the mean. 3, In four continued proportionals, the sum of the extremes is greater than the sum of the two means. 4, In any two quantities, one-fourth the square of the sum of the cubes, is greater than the cube of the product of the two quantities. 5, Of any two quantities q and r, then $\frac{1}{27}(qq+qr+rr)^3 > \frac{1}{4}(qqr+qrr)^2$. 6, If any quantity be divided into three unequal parts, the square of $\frac{1}{3}$ of the whole quantity is greater than $\frac{1}{3}$ of the sum of the three products made of the three unequal parts. 7, Also the cube of the $\frac{1}{3}$ part of the

whole, is greater than the solid or continual product of the three unequal parts.

Sect. 6. Æquationum communium reductio per gradus alicujus parodici exclusionem & radicis supposititiæ mutationem. Here are a great many examples of reducing and transforming equations of the 2d, 3d, and 4th degrees; chiefly either by multiplying the roots of equations in any proportion, as was done by Vieta, or increasing or diminishing the root by a given quantity, after the manner of Cardan. The former of these reductions is performed by multiplying the terms of the equation by the corresponding terms of a geometrical progression, the first term being 1, and the 2d term the quantity by which the root is to be multiplied. And the other reduction, or transforming to another root, which may be greater or less than the given root by a given quantity, is performed commonly by substituting $e +$ or $- b$ for the given root a, by which the equation is reduced to a simpler form. Other modes of substitution are also used; one of which is this, viz, substituting $\frac{ee \pm bb}{e}$ or $e \pm \frac{bb}{e}$ for the root a in the given equation $aaa \mp 3.bba = 2.ccc$, by which it reduces to this quadratic form $e^6 \mp 2c^3 e^3 = - b^6$, from which Cardan's forms are immediately deduced; namely $e = \sqrt{3)}\, c^3 - \sqrt{c^6 \pm b^6}$, and therefore a or $e \pm \frac{bb}{e} =$ $\sqrt{3)}\, c^3 + \sqrt{c^6 \pm b^6} \pm \sqrt{3)}\, c^3 - \sqrt{c^6 \pm b^6}$; where he denotes the cubic or 3d root thus $\sqrt{3)}$, but without any vinculum over the compound quantities.

In this section, Harriot makes various remarks as they occur: thus, he remarks, and demonstrates, that $eee - 3.bbe = - cce$

$- 2.bbb$ is an impossible equation, or has no affirmative root. He remarks also that the three cases of the equation $aaa - 3.bba = + 2.ccc$ are similar to the three conic sections; namely to the hyperbola when $c > b$, to the parabola when $c = b$, or to the ellipsis when $c < b$, and for which reason this case is not generally resolvable in species.

Having thus shown how to simplify equations, and prepare them for solution, Harriot enters next upon the second part of his work, being the

Exegetice Numerosa,

or the numeral resolution of all sorts of equations by a general method, which is exemplified in a great number of equations, both simple and affected, as far as the 5th power inclusive; and they are commonly prepared, by the foregoing parts, by freeing them from their 2d term, &c. These extractions are explained and performed in a way different from that of Vieta; and the examples are first in perfect or terminate roots, and afterwards for irrational or interminate ones, to which Harriot approximates, by adding always periods of ciphers to the given number or resolvend, as far as necessary in decimals, which are continued and set down as such, but with their proper denominator 10, or 100, or 1000, &c.

He then concludes the work with

Canones Directorii,

which form a collection of the cases or theorems for making the foregoing numeral extractions, ready arranged for use, under the various forms of equations, with the factors necessary to form the several resolvends and subtrahends.

And from a review of the whole work, it appears that Harriot's inventions, peculiarities, and improvements in algebra, may be comprized in the following particulars.

1st. He introduced the uniform use of the small letters, *a*, *b*, *c*, *d*, &c, viz, the vowels *a*, *e*, &c, for unknown quantities, and the consonants *b*, *c*, *d*, *f*, &c, for the known ones; which he joins together like the letters of a word, to represent the multiplication or product of any number of these literal quantities, and prefixing the numeral coefficient as we do at present, except only separated by a point, thus 5.*bbc*. For a root, he set the index of the root after the mark

$\sqrt{}$; as $\sqrt{3}$) for the cube root. He also introduced the characters $>$ and $<$, for greater and less; and in the reduction of equations, he arranged the operations in separate steps or lines, setting the explanations in the margin on the left hand, for each line. By which, and other means, he may be considered as the introducer of the modern state of Algebra, which quite changed its form under his hands.

2d. He showed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones, or binomial roots; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; and from which many of the most important properties have since been deduced.

3d. He greatly improved the numeral exegesis, or extraction of the roots of all equations, by clear and explicit rules and methods, drawn from the foregoing generation or composition of affected equations of all degrees.

OF OUGHTRED'S CLAVIS.

Outred was contemporary with Harriot, but lived a long time after him, as he died only in 1660, at 87 years of age. His Clavis was first published in 1631, the same year in which Harriot's Algebra was published by his friend Warner. In this work, Oughtred chiefly follows Vieta, in the notation by the capitals A, B, C, D, &c, in the designation of products, powers, and roots, though with some few variations. His work may be comprehended under the following particulars.

1. *Notation.* This extends to both Algebra and Arithmetic, vulgar and decimal. The Algebra chiefly after the manner of Vieta, as abovesaid. And he separates the decimals from the integers thus, $21\underline{|56}$, which is the first time I have observed such a separation, and the decimals set down without their denominator.

2. The common rules or operations of Arithmetic and Algebra. In algebraic multiplication, he either joins the letters together like a word, or connects them by the mark $\times$, which is the first introduction of this character of multiplication: thus A $\times$ A or AA or A*q*. But omitting the vinculum over compound factors, used by Vieta. He introduces here many neat and useful contractions in multiplication and division of decimals: as that common one of inverting the multiplier, to have fewer decimals, and abridge the work; that of omitting always one figure at a time, of the divisor, for the same purpose; dividing by the component factors of a number, instead of the number itself; as 4 and 6 for 24; and many other neat contractions. He states his proportions thus 7.9 :: 28.36, and denotes continued proportion thus $\div\!\!:$; which is the first time I have observed these characters.

3. Invents and describes various symbolical marks or abbreviations, which are not now used.

4. *The genesis and analysis of powers.* Denotes powers like Vieta, and also roots, thus $\sqrt{q6}$, $\sqrt{c20}$, $\sqrt{qq24}$, &c; and much in his manner too performs the numeral extraction of roots. He here gives a table of the powers of the binomial A + E as far as the 10th power, with all their terms and coefficients, or unciæ as he calls them, after Vieta.

5. *Equations.* He here gives express and particular directions for the several sorts of reductions, according as the form of the equation may require. He uses the letter *u* after $\sqrt{}$, for universal, instead of the vinculum of Vieta. And he observes that the signs of all the terms of the powers of A + E are positive, but those of A − E are alternately positive and negative.

6. Next follow many properties of triangles and other geometrical figures; and the first instance of applying Algebra to Geometry, so as to investigate new geometrical properties; also, after the algebraical resolution of each problem, he commonly deduces and gives a geometrical

construction adapted to it. He gives also a good tract on angular sections.

7. The work concludes with the numeral solution of affected equations, in which he follows the manner of Vieta, though he is more explicit.

OF DESCARTES.

Descartes's Geometry was first published in 1637, being six years after the publication of Harriot's Algebra. That work was rather an application of Algebra to Geometry, than the science either of Algebra or Geometry itself, purely and properly so called. And yet he made improvements in both. We must observe, however, that all the properties of equations, &c, which he sets down, are not to be considered as even meant by himself for new inventions or discoveries; but as statements and enumerations of properties, before known and taught by other authors, which he is about to make some use or application of, and for which reason it is that he mentions those properties.

Descartes's Geometry consists of three books. The first of these is, *De Problematibus, quæ construi possunt, adhibendo tantum rectas lineas & circulos.* He here accommodates or performs arithmetical operations by Geometry, supposing some line to represent unity, and then, by means of proportionals, showing how to multiply, divide, and extract roots, by lines. He next describes the notation he uses, but not because it is a new one, for it is the same as had been used by former authors, viz, $a + b$ for the addition of a and b, also $a - b$ for their subtraction, ab multiplication, $\frac{a}{b}$ division, aa or a^2 the square of a, a^3 its cube, &c: also $\sqrt{a^2 + b^2}$ for the square root of $a^2 + b^2$, and $\sqrt{c.a^3 - b^3 + abb}$ for the cube root, &c, with a bar over the quantities. He then observes, after Stifelius, that there must be as many equations as there are unknown lines or quantities; and that they must be reduced all

to one final equation, by exterminating all the unknown letters except one; when the final equation will appear like these,

$z \propto b$, or

$z^2 \propto -az + b^2$, or

$z^3 \propto +az^2 + b^2z - c^3$, or

$z^4 \propto +az^3 + b^2z^2 - c^3z + d^4$, &c.

Where he uses $\propto$ for $=$ or equality, setting the highest term or power alone on one side of the equation, and all the other terms on the other side, with their proper signs.

Descartes next defines plane problems, namely, such as can be resolved by right lines and circles, described on a plane superficies; and then the final equation rises only to the 2d power of the unknown letter. He then constructs such equations, viz, quadratics, by the circle, thus finding geometrically the root or roots, that is, the positive ones. But when the lines, by which the roots are determined, neither cut nor touch, he observes that the equation has then no possible root, or that the problem is impossible. He then concludes this book with the algebraical solution of the celebrated problem, before treated of by the antients, namely, to find a point, or the locus of all the points, from which a line being drawn, to meet any number of given lines in given angles, the product of the segments of some of them shall have a given ratio to that of the rest.

Lib. 2. De Natura Linearum Curvarum. This is a good algebraical treatise on curve lines in general, and the first of the kind that has been produced by the moderns. Here the nature of the curve is expressed by an equation containing two unknown or variable lines, and others that are known or constant, as $y^2 \propto cy - \frac{cxy}{b} + ay - ac$. But, not relating to pure Algebra, the particulars will be most properly placed under the article of curve lines, and other terms relating to them. Only one discovery, among many ingenious applications of Algebra to Geometry, may here be particularly noticed, as it may be considered as the first step towards the arithmetic of infinites; and that is the

method of tangents, here given, or, which comes to the same thing, of drawing a line perpendicular to a curve at any point; which is an ingenious application of the general form of an equation, generated in Harriot's way, that has two equal roots, to the equation of the curve. Of which a particular account will be given at the article TANGENTS.

Lib. 3. *De Constructione Problematum Solidorum, et Solida excedentium.* Descartes begins this book with remarks on the nature and roots of equations, observing, that they have as many roots as dimensions, which he shows, after Harriot, by multiplying a certain number of simple binomial equations together, as $x - 2 \propto 0$, and $x - 3 \propto 0$, and $x - 4 \propto 0$, producing $x^3 - 9xx + 26x - 24 \propto 0$. He here remarks that equations may sometimes have their roots *false*, or what we call negative, which he opposes to those that are positive, or as he calls them *true*, as Cardan had done before. As a natural deduction from the generation or composition of equations, by multiplication, he infers their resolution, or depression, or decomposition, namely, dividing them by the binomial factors which were multiplied to produce the equation: and he observes, that, by this operation it is known that this divisor is one of the binomial roots, and that there can be no more roots than dimensions, or than those which form with the unknown letter x, binomials that will exactly divide the equation, as Harriot had shown before.

Descartes adverts to several other properties, mostly known before, which he has occasion to make use of in the progress of his work; such as, that equations may have as many true roots as the terms have changes of the signs + and −, and as many false ones as successions of the same signs; which number and nature of the roots had before been partly shown by Cardan and Vieta, from the relation of the coefficients, and their signs, and the same thing appears more fully by Harriot's 5th section. And hence Descartes infers the method of changing the true roots to false, and the false to true, namely, by changing the signs of the

even terms only, as Cardan had taught before. Descartes then adverts to other reductions and transmutations which had been taught by Cardan, Vieta, and Harriot, such as, To increase or diminish the roots by any quantity; To take away the 2d term: To alter the roots in any proportion, and thence to free the equation from fractions and radicals.

Descartes next remarks that the roots of equations, whether true or false, may be either real or imaginary; as in the equation $x^3 - 6xx + 13x - 10 \propto 0$, which has only one real root, namely 2. The imaginary roots were first noticed by Albert Girard, as before mentioned. He then treats of the depression of a cubic equation to a quadratic, or plane problem, that it may be constructed by the circle, by dividing it by some one of the binomial factors, which, in Harriot's way, compose the equation. Peletarius having shown that the simple root is one of the divisors of the known term of the equation, and Harriot that that term, is the continual product of all the roots, Descartes therefore tries all the simple divisors of that term, till he finds one of them which, connected with the unknown letter x, by $+$ or $-$, will exactly divide the equation. And the process is the same for higher powers than the cube. But when a divisor cannot be thus found, for depressing a biquadratic equation to a cubic, he gives another rule, which is a new one, for dissolving it into two quadratics, by means of a cubic equation, in this manner:

Let the given biqu. be $+ x^4 * \cdot pxx \cdot qx \cdot r \propto 0$;

Which suppose equal to the product of these two $\left\{\begin{array}{l} + xx - yx + \frac{1}{2}yy \cdot \frac{1}{2}p \cdot \frac{q}{2y} \propto 0, \\ + xx + yx + \frac{1}{2}yy \cdot \frac{1}{2}p \cdot \frac{q}{2y} \propto 0; \end{array}\right.$

where the sign of $\frac{1}{2}p$ in the two quadratics must be the same as the sign of p in the given equation, and in the 1st quadratic the sign of $\frac{q}{2y}$ must be the same as the sign of q, but in the 2d quadratic the contrary. Then if there be found the root yy of this cubic equation $y^6 \cdot 2py^4 + \begin{smallmatrix}+pp\\ \cdot 4r\end{smallmatrix}\, yy - qq \propto 0$,

where the sign $2p$ is the same as of p in the given biquadratic, but the sign of $4r$ contrary to that of r in the same: Then the value of y, hence deduced, being substituted for it in the two quadratic equations, and their two pairs of roots taken, they will be the four roots of the proposed biquadratic. And thus also, he hints, may equations of the 6th power be reduced to those of the 5th, and those of the 8th power to those of the 7th, and so on. Descartes does not give the investigation of this rule; but it has evidently been done, by assuming indeterminate quantities, after the manner of Ferrari and Cardan, as coefficients of the terms of the two quadratic equations, and after multiplying the two together, determining their values by comparing the resulting terms with those of the proposed biquadratic equation.

After these reductions, which are only mentioned for the sake of the geometrical constructions which follow, by simplifying and depressing the equations as much as they will admit, Descartes then gives the construction of solid and other higher problems, or of cubic and higher equations, by means of parabolas and circles; where he observes, that the false roots are denoted by the ordinates to the parabola lying on the contrary side of the axis to the true roots, as before mentioned by Girard. Finally, these constructions are illustrated by various problems concerning the trisecting of an angle, and the finding of two or four mean proportionals; which concludes this ingenious work.

From the foregoing analysis may easily be collected the real inventions and improvements made in algebra by Descartes. His work, as has been observed before, is not algebra itself, but the application of algebra to geometry, and the algebraical doctrine of curve lines, expressing and explaining their nature by algebraical equations; and, on the contrary, constructing and explaining equations by means of the curve lines. What respects the geometrical parts of this tract we shall have occasion to advert to elsewhere; and therefore shall here only enumerate the circumstances which

belong more peculiarly to the science of Algebra, which I shall distinguish into the two heads, of improvements, and inventions. And

1st. Of his improvements.—That he might fit equations the better for their application in the construction of problems, Descartes mentions, as it were by-the-by, many things concerning the nature and reduction of equations, without troubling himself about the first inventors of them, stating them in his own terms and manner, which is commonly more clear and explicit, and often with improvements of his own. And under this head we find that he chiefly followed Cardan, Vieta, and Harriot, but especially the last, and explains some of their rules and discoveries more distinctly, and varies but a little in the notation, putting the first letters of the alphabet for the known, and the latter letters for the unknown quantities; also x^3 for *aaa*, &c; and ∞ for =. But Herigone used the numeral exponents in the same manner some years before, as well as several older authors. Descartes explained or improved most parts of the reductions of equations, in their various transmutations, the number and nature of their roots, true and false, real and what he denominates imaginary, called involved by Girard; and the depression of equations to lower degrees:

2d. As to his inventions and discoveries in algebra, they may be comprized in these particulars: namely, the application of algebra to the geometry of curve lines, the constructing equations of the higher orders, and a rule for resolving biquadratic equations by means of a cubic and two quadratics.

Having now traced the science of Algebra from its origin and rude state, down to its modern and more polished form, in which it has ever since continued, with very little variation; having analysed all or most of the principal authors, in a chronological order, and deduced the inventions and improvements made by each of them; from this time the authors both become too numerous, and their improvements too inconsiderable, to merit a detail in the same minute and

circumstantial way: and besides, these will be better explained in a particular manner under the word or article to which each of them severally belongs. It may therefore now suffice to enumerate, or announce only in a cursory way, the chief improvements and authors on algebra down to the present time.

After the publication of the Geometry of Descartes, a great many other ingenious men followed the same course, applying themselves to algebra and the new geometry, to the mutual improvement of them both; which was done chiefly by reasoning on the nature and forms of equations, as generated and composed by Harriot. Before proceeding upon these, however, it is but proper to take notice here of Fermat, a learned and ingenious mathematician, who was contemporary and a competitor of Descartes, for his brightest discoveries, which he was in possession of before the geometry of Descartes appeared. Namely, the application of algebra to curve lines, which he expressed by an algebraical equation, and by them constructed equations of the 3d and 4th orders; also a method of tangents, and a method de maximis et minimis, which approach very near to he method of Fluxions or Increments, which they strikingly resemble, both in the manner of treating the problems, and in the algebraic notation and process. The particulars of which, see under the proper heads. Besides these, Fermat was deeply learned in the Diophantine problems, and the best edition of Diophantus's Arithmetic, is that which contains the notes of Fermat on that ingenious work.

But to return to the successors of Descartes. His geometry having been published in Holland, several learned and ingenious mathematicians, of that country, presently applied themselves to cultivate and improve it; as Schooten, Hudde, Van-Heuraet, De Witte, Slusius, Huygens, &c; besides M. de Beaune, and perhaps some others in France.

Francis Schooten, professor of mathematics in the university of Leyden, was one of the first cultivators of the new

geometry. He translated Descartes's Geometry out of French into Latin, and published it in 1649, with his commentary on it, as also Brief Notes of M. de Beaune; both of them containing many ingenious and useful things. And in 1659 he gave a new edition of the same, in two volumes, with the addition of several other ingenious pieces: as, two posthumous tracts of de Beaune, the one on the nature and constitution, the other on the limits of equations, showing how to assign the limits between which are contained the greatest and least roots of equations, extended and completed by Erasmus Bartholine: two letters of M. Hudde on the reduction of equations, and on the maxima and minima of quantities, containing many ingenious rules; among which are some concerning the drawing of tangents, and on the equal roots of equations, which he determines by multiplying the terms of the equation by the terms of any arithmetical progression, 0 being one of the terms, the equation is commonly depressed one degree lower; also a tract of Van Heuraet on the rectification of curve lines; the elements of curves by De Witte; Schooten's principles of universal mathematics, or introduction to Descartes's geometry, which had before been published by itself in 1651; and to the end of the work is added a posthumous piece of Schooten's (for he died while the 2d vol. was printing) intitled *Tractatus de concinnandis demonstrationibus geometricis ex calculo algebraico*. Schooten also published, in 1657, *Exercitationes Mathematicæ*, in which are contained many curious algebraical and analytical pieces, among others of a geometrical nature.

An elaborate commentary on Descartes's Geometry was also published by F. Rabuel, a Jesuit; and James Bernoulli, enriched with notes, an edition of the same, printed at Basil in 169—.

The celebrated Huygens also, among his great discoveries, very much cultivated the algebraical analysis: and he is often cited by Schooten, who relates divers inventions of his, while he was his pupil.

Slusius, a canon of Liege, published in 1659, *Mesolabum, seu duæ mediæ propor. per circulum & ellips. vel hyperb. infinitis modis exhibitæ*; by which, any solid problem may be constructed by infinite different ways. And in 1668, he gave a second edition of the same, with the addition of the analysis, and a miscellaneous collection of curious and important problems, relating to spirals, centres of gravity, maxima and minima, points of inflexion, and some Diophantine problems; all showing him deeply skilled in Algebra and Geometry.

There have been a great number of other writers and improvers of Algebra, of which it may suffice slightly to mention the chief part, as in the following catalogue.

In 1619 several pieces of Van Collen, or Ceulen, were translated out of Dutch into Latin, and published at Leyden by W. Snell; among which are contained a particular treatise on surds, and his proportion of the circumference of a circle, to its diameter.

In 1621 Bachet published, in Greek and Latin, an edition of Diophantus, with many notes. And another edition of the same was published in 1670, with additions by Fermat.

In 1624 Bachet's *Problemes Plaisans et Delectables*, being curious problems in mathematical recreations.

In 1634 Herigone published, at Paris, the first course of mathematics, in 5 vols. 8vo; in the 2d of which is contained a good treatise on Algebra; in which he uses the notation by small letters, introduced by the Algebra of Harriot, which was published three years before, though the rest of it does not resemble that work, and one would suspect that Herigone had not seen it. The whole of this piece bears evident marks of originality and ingenuity. Besides + for *plus*, he uses ∽ for *minus*, and | for *equality*, with several other useful abbreviations and marks of his own. In the notation of powers, he does not repeat the letters like Harriot, but subjoins the numeral exponents, to the letter, as Descartes did two years afterwards. And Herigone uses

the same numeral exponents for roots, as $\sqrt{3}$ for the cube root.

In 1635 Cavalerius published his *Indivisibles;* which proved a new æra in analytics, and gave rise to other new modes of computation in analytics.

About 1640, et seq. Roberval made several notable improvements in analytics, which are published in the early volumes of the Memoirs of the Academy of Sciences; as, 1. A tract on the composition of motion, and a method of tangents. 2, *De recognitione æquationum.* 3, *De geometrica planarum & cubicarum æquationum resolutione.* 4, A treatise on indivisibles, &c.

In 1643 De Billy published *Nova Geometriæ Clavis Algebra.* And in 1670 *Diophantus Redivivus.* He was an author particularly well skilled in Diophantine problems.

In 1644 Renaldine published, in 4to, *Opus Algebraicum,* both ancient and modern, with mathematical resolution and composition. And in 1665, in folio, the same, greatly enlarged, or rather a new work, which is very heavy and tedious. In this work Renaldine uses the parentheses (a^2+b^2) as a vinculum, instead of the line over, as $\overline{a^2+b^2}$.

In 1655 was published Wallis's *Arithmetica Infinitorum,* being a new method of reasoning on quantities, or a great improvement on the Indivisibles of Cavalerius, and which in a great measure led the way to infinite series, the binomial theorem, and the method of fluxions. Wallis here treats ingeniously of quadratures and many other problems, and gives the first expression for the quadrature of the circle by an infinite series. Another series is here added for the same purpose, by the Lord Brouncker, the first president of the Royal Society.

In 1659 was published *Algebra Rhonii Germanice;* which was in 1668, translated into English by Mr. Thomas Brancker, with additions and alterations by Dr. John Pell.

In 1661 was published in Dutch, a neat piece of Algebra by Mr. Kinckhuysen; which Sir I. Newton, while he was professor of mathematics at Cambridge, made use of and

improved, and he meant to republish it, with the addition of his method of fluxions and infinite series; but was prevented by the accidental burning of some of his papers.

In 1665 or 1666 Sir Isaac Newton made several of his brightest discoveries, though they were not published till afterwards: such as the binomial theorem; the method of fluxions and infinite series; the quadrature, rectification, &c of curves; to find the roots of all sorts of equations, both numeral and literal, in infinite converging series; the reversion of series, &c. Of each of which a particular account may be seen in their proper articles.

In 1666 M. Frenicle gave several curious tracts concerning combinations, magic squares, triangular numbers, &c; which were printed in the early volumes of the Memoirs of the Academy of Sciences.

In 1668 Thomas Brancker published a translation of Rhonius's Algebra, with many additions by Dr. John Pell, who used a peculiar method of registering the steps in any algebraical process, by means of marks and abbreviations in a small column drawn down the margin, by which each line, or step, is clearly explained, as was before done by Harriot in words at length.

In 1668 Mercator published his *Logarithmotechnia*, or method of constructing logarithms; in which he gives the quadrature of the hyperbola, by means of an infinite series of algebraical terms, found by dividing a simple algebraic quantity by a compound one, and for the first time that this operation was given to the public, though Newton had before that expanded all sorts of compound algebraical quantities into infinite series.

In the same year was published James Gregory's *Exercitationes Geometricæ*, containing, among other things, a demonstration of Mercator's quadrature of the hyperbola, by the same series.

And in the same year was published, in the Philosophical Transactions, Lord Brouncker's quadrature of the hyperbola, by another infinite series of simple rational terms,

which he had been in possession of since the year 1657, when it was announced to the public by Dr. Wallis. Lord Brouncker's series for the quadrature of the circle, had been published by Wallis in his Arithmetic of Infinites, as before mentioned.

In 1669 Dr. Isaac Barrow published his Optical and Geometrical Lectures, abounding with profound researches on the dimensions and properties of curve lines; but particularly to be noticed here for his method of tangents, by a mode of calculation similar to that of Fluxions, or Increments, from which these differ but little, except in the notation. Of these lectures, the 13th merits the most special notice, being entirely employed on equations, and delivered in a very curious way. He there treats of the nature and number of their roots, and the limits of their magnitudes, from the description of lines accommodated to each, viz, treating the subject as a branch of the doctrine of maxima and minima, which in the opinion of some persons, is the right way of considering them, and as far preferable to the so much boasted invention of the generation of equations from each other, explained by Harriot and Descartes.

In 1673 was published, in 2 vols. folio, *Elements of Algebra*, by John Kersey; a very ample and complete work, in which Diophantus's problems are fully explained.

In 1675 were published Nouveaux Elemens des Mathematiques, par J. Prestet, prêtre: a prolix and tedious work, which he presumptuously dedicated to God Almighty.

About 1677 Leibnitz discovered his *Methodus Differentialis*, or else made a variation in Newton's Fluxions, or an extension of Barrow's method, for it is not certain which. He gave the first instance of it in the Leipsic Acts for the year 1684. He also improved infinite series, and gave a simple one for the quadrature of the circle, in the same acts for 1682.

In 1682 Ismael Bulliald published, in folio, his *Opus Novum ad Arithmeticam Infinitorum*, being a large amplification of Wallis's Arithmetic of Infinites.

In 1683 Tschirnausen gave a memoir, in the Leipsic Acts, concerning the extraction of the roots of all equations in a general way; in which he promised too much, as the method did not succeed.

In 1684 came out, in English and Latin, 4to, Thomas Baker's *Geometrical Key, or Gate of Equations Unlock'd;* being an improvement of Descartes's construction of all equations under the 5th degree, by means of a circle and only one and the same parabola for all equations, using any diameter instead of the axis of the parabola.

In 1685 was published, in folio, Wallis's *Treatise of Algebra, both Historical and Practical,* with the addition of several other pieces; exhibiting the origin, progress, and advancement of that science, from time to time. It cannot be denied that, in this work, Wallis has shown too much partiality to the Algebra of Harriot. Yet, on the other hand, it is as true, that M. de Gua, in his account of it, in the Memoirs of the Academy of Sciences for 1741, has run at least as far into the like extreme on the contrary side, with respect to the discoveries of Vieta; and both these I believe from the same cause, namely, the want of examining the works of all former writers on Algebra, and specifying their several discoveries; as has been done in the course of this tract.

In 1687 Dr. Halley gave, in the Philos. Trans. the construction of cubic and biquadratic equations, by a parabola and circle; with improvements on what had been done by Descartes, Baker, &c. Also, in the same Transactions, a memoir on the number of the roots of equations, with their limits and signs.

In 1690 was published, in 4to, by M. Rolle, *Traité d'Algébre;* in 1699 *Une Methode pour la Resolution des Problemes indeterminés;* and in 1704 *Memoires sur l'inverse des tangents;* and other pieces.

In 1690 Joseph Raphson published *Analysis Æquationum Universalis;* being a general method of approximating to the roots of equations in numbers. And in 1715 he published the *History of Fluxions,* both in English and Latin.

In 1690 was also published, in 4 vols 4to, Dechale's *Cursus seu mundus mathematicus*; in which is a piece of algebra, of a very old-fashioned sort, considering the time when it was written.

About 1692, and at different times afterwards, De Lagny published many pieces on the resolution of equations in numbers, with many theorems and rules for that purpose.

In 1693 was published, in a neat little volume, *Synopsis Algebraica, opus posthumum Johannis Alexandri.*

In 1694, Dr. Halley gave, in the Philos. Trans. an ingenious tract on the numeral extraction of all roots, without any previous reduction. And this tract is also added to some editions of Newton's Universal Arithmetic.

In 1695 Mr. John Ward, of Chester, published, in 8vo, *A Compendium of Algebra*, containing plain, easy, and concise rules, with examples in an easy and clear way. And in 1706 he published the first edition of his *Young Mathematician's Guide*, or a plain and easy introduction to the mathematics: a book which is still in great request, especially with beginners, and which has been ever since the ordinary introduction of the greatest part of the mathematicians of this country.

In 1696 the Marquis de l'Hôpital published his *Analyse des infiniment petits.* And gave several papers to the Leipsic Acts and the Memoirs of the Academy of Sciences. He left behind him also an ingenious treatise, which was published in 1707, intitled *Traité analytique des Sections Coniques, et de la construction des lieux geometriques.*

In 1697, and several other years, Mr. Ab. Demoivre gave various papers in the Philos. Trans. containing improvements in Algebra: viz, in 1697, a method of raising an infinite multinomial to any power, or extracting any root of the same. In 1698, The extraction of the root of an infinite equation. In 1707, Analytical solution of certain equations of the 3d, 5th, 7th, &c degree. In 1722, Of algebraic fractions and recurring series. In 1738, The reduction of radicals into more simple forms. Also in 1730, he published

Miscellanea analytica de seriebus & quadraturis, containing great improvements in series, &c.

In 1698, Mr. Richard Sault published, in 4to, *A New Treatise of Algebra, apply'd to numeral questions and geometry. With a converging series for all manner of Adfected Equations.* The series here alluded to, is Mr. Raphson's method of approximation, which had been lately published.

In 1699 Hyac. Christopher published at Naples, in 4to, *De constructione æquationum.*

In 1702 was published Ozanam's Algebra; which is chiefly remarkable for the Diophantine analysis. He had published his mathematical dictionary in 1691, and in 1693 his course of mathematics, in 5 vols. 8vo, containing also a piece on algebra.

In 1704, Dr. John Harris published his *Lexicon Technicum*, the first dictionary of arts and sciences: a very plain and useful book, especially in the mathematical articles. And in 1705 a neat little piece on algebra and fluxions.

In 1705 M. Guisnée published, in 4to, his *Application de l'algebre a la geometrie:* a useful book.

In 1706 Mr. William Jones published his *Synopsis Palmariorum Mathescos*, or a new introduction to the mathematics: a very useful compendium in the mathematical sciences. And in 1711 he published, in 4to, a collection of Sir Isaac Newton's papers, intitled *Analysis per quantitatum series, fluxiones, ac differentias: cum enumeratione linearum tertii ordinis.*

In 1707 was published, by Mr. Whiston, the first edition of Sir Isaac Newton's *Arithmetica Universalis: sive de compositione et resolutione arithmetica liber:* and many editions have been published since. This work it seems was the text book used by our great author in his lectures, while he was professor of mathematics in the university of Cambridge. And though it was never intended for publication, it contains many and great improvements in analytics; particularly in the nature and transmutation of equations; the limits of the roots of equations; the number of impossible roots; the

invention of divisors, both surd and rational; the resolution of problems, arithmetical and geometrical; the linear construction of equations; approximating to the roots of all equations, &c. To the later editions of the book is commonly subjoined Dr. Halley's method of finding the roots of equations. As the principal parts of this work are not adapted to the circumstance of beginners, there have been published commentaries upon it by several persons, as s'Gravesande, Castilion, Wilder, &c.

In 1708 M. Reyneau published his *Analyse Demontrée*, in 2 vols. 4to. And in 1714 *La Science du Calcul*, &c.

In 1709 was published an English translation of Alexander's algebra. With an ingenious appendix by Humphry Ditton.

In 1715 Dr. Brooke Taylor published his *Methodus Incrementorum:* an ingenious and learned work. And in the Philos. Trans. for 1718, An improvement of the method of approximating to the roots of equations in numbers.

In 1717 M. Nicole gave, in the memoirs of the academy of sciences, a tract on the calculation of finite differences. And in several following years, he gave various other tracts on the same subject, and on the resolution of equations of the 3d degree, and particularly on the irreducible case in cubic equations.

Also in 1717 was published a treatise on Algebra by Philip Ronayne.

Also in 1717 Mr. James Sterling published *Lineæ tertii Ordinis;* an ingenious work, containing good improvements in analytics. And in 1730 *Methodus Differentialis: sive tractatus de summatione et interpolatione serierum infinitarum:* with great improvements on infinite series.

In 1726 and 1729 Maclaurin gave, in the Philos. Trans. tracts on the imaginary roots of equations. And afterwards was published, from his posthumous papers, his treatise on Algebra, with its application to curve lines.

In 1727 came out s'Gravesand's Algebra, with a specimen of a commentary on Newton's universal arithmetic.

In 1728 Mr. Campbell gave, in the Philos. Trans. an ingenious paper on the number of impossible roots of equations.

In 1732 was published Wolfius's Algebra, in his course of mathematics, in 5 vols. 4to.

In 1735 Mr. John Kirby published his arithmetic and algebra. And in 1748 his doctrine of ultimators.

In 1740 were published Mr. Thomas Simpson's Essays; in 1743 his Dissertations, and in 1757 his Tracts; in all which are contained several improvements in series and other parts of Algebra. As also in his algebra, first printed in 1745, and in his Select Exercises, in 1752.

Also in 1740 was published professor Saunderson's Elements of Algebra, in 2 vols. 4to.

In 1741 M. de la Caille published *leçons de mathematiques; ou elemens d' algebre & de geometrie.*

Also in 1741, in the memoirs of the academy of sciences, were given two articles by M. de Gua, on the number of positive, negative, and imaginary roots of equations. With an historical account of the improvements in Algebra; in which he severely censures Wallis for his partiality; a circumstance in which he himself is not less faulty.

In 1746 M. Clairaut published his Elemens d'algebre, in which are contained several improvements, especially on the irreducible case in cubic equations. He has also several good papers on different parts of analytics, in the memoirs of the academy of sciences.

In 1747 M. Fontaine gave, in the memoirs of the academy of sciences, a paper on the resolution of equations. Besides some analytical papers in the memoirs of other years.

In 1761 M. Castillion published, in 2 vols 4to, Newton's universal arithmetic, with a large commentary.

In 1763 Mr. Emerson published his Increments. In 1764 his Algebra, &c.

In 1764 Mr. Landen published his Residual Analysis. In 1765 his Mathematical Lucubrations. And in 1780 his Ma-

thematical Memoirs. All containing good improvements in infinite series, &c.

In 1770 was published, in the German language, Elements of Algebra by M. Euler. And in 1774 a French translation of the same. The memoirs of Berlin and Petersburg academies also abound with various improvements in series and other branches of analysis by this great man.

In 1775 was published at Bologna, in 2 vols. 4to, *Compendio d'Analisi di* Girolamo Saladini.

In 1797 was published, by Signor Cossali, in the Italian language, in 2 vols. 4to, a very elaborate history of the origin of the first introduction of algebra into Italy, and of its progress and more early improvements among his countrymen. In this work, the author clearly shows, what however was well known long before, that the science was imported into Italy from the Arabians; but he seems erroneously to ascribe the very invention of algebra to these latter people, from his ignorance of the late discoveries lately made of the writings on this science among the Hindus.

Besides the foregoing, there have been many other authors who have given treatises on Algebra, or who have made improvements on series and other parts of Algebra; as Schonerus, Coignet, Salignat, Laloubere, Hemischius, Degraave, Mescher, Henischius, Roberval, the Bernoullis, Malbranche, Agnesi, Wells, Dodson, Manfredi, Begnault, Bezout, Bossut, Rowning, Maseres, Waring, Lorgna, de la Grange, de la Place, Bertrand, Kuhnius, Hales, and many other authors. Dr. Waring and the Rev. M. Vince, of Cambridge, have both given many improvements and discoveries in series, and in other branches of analysis. Those of Mr. Vince are chiefly contained in the latter volumes of the Philos. Trans.; where also are several of Dr. Waring's; but the bulk of this gentleman's improvements are contained in his separate publications, particularly the *Meditationes Algebraicæ* published in 1770; the *Proprietates Algebraicarum Curvarum*, in 1772; and the *Meditationes Analyticæ*, in 1776.

TRACT XXXIV.

NEW EXPERIMENTS IN GUNNERY; FOR DETERMINING THE FORCE OF FIRED GUNPOWDER, THE INITIAL VELOCITY OF CANNON BALLS, THE RANGES OF PROJECTILES AT DIFFERENT ELEVATIONS, THE RESISTANCE OF THE AIR TO PROJECTILES, THE EFFECT OF DIFFERENT LENGTHS OF GUNS, AND OF DIFFERENT QUANTITIES OF POWDER, &c, &c.

Sect. 1.

At Woolwich, in the year 1775, in conjunction with some able officers of the Royal Regiment of Artillery, and other ingenious gentlemen, was first instituted a course of experiments on fired gunpowder and cannon balls, similar to the present course. My account of them was presented to the Royal Society, who honoured it with the gift of the annual gold medal, and printed it in the Philosophical Transactions for the year 1778.* The object of those experiments, was, the determination of the actual velocities with which balls are impelled from given pieces of cannon, when fired with given charges of powder. They were made according to

* The public delivery of the medal to Dr. H. and the pronouncing of an excellent appropriate oration on that occasion, on the 30th of November 1778, to the largest and most respectable audience that ever attended such a meeting of the Royal Society, by their excellent president Sir John Pringle, was the last act of that gentleman in his official capacity: his delivery of the medal, and stepping out of the chair for the last time, occurring in the same moment.—He was immediately succeeded by Sir Joseph Banks.

The paper itself contained the first part of a series of military experiments, then projected, and conducted through many succeeding years. An account of some more of these annual experiments was given in my 4to. volume of Tracts, published in 1786. The substance of those publications is now delivered in a more condensed form, and connected with the continuation and conclusion of the experiments, being the more important part of the same, accompanied with deductions from the whole, tending to render them useful to the purposes of natural philosophy in general, and of the artillery practice in particular.

the method invented by the very ingenious Mr. Robins, and described in his treatise on the New Principles of Gunnery, of which an account was printed in the Philosophical Transactions for the year 1743. Before the discoveries and inventions of that gentleman, very little progress had been made in the true theory of military projectiles. His book however contained such important discoveries, that it was soon translated into several of the languages on the continent, and the late famous Mr. L. Euler honoured it with a very learned and extensive commentary, in his translation of it into the German language. That part of Mr. Robins's book has always been much admired, which relates to the experimental method of ascertaining the actual velocities of shot, and in imitation of which, but on a large scale, those experiments were made which were described in my paper. Experiments in the manner of Mr. Robins were generally repeated by his commentators, and others, with universal satisfaction; the method being so just in theory, so simple in practice, and altogether so ingenious, that it immediately gave the fullest conviction of its excellence, and the eminent abilities of the inventor. The use which our author made of his invention, was to obtain the real velocities of bullets experimentally, that he might compare them with those which he had computed *a priori* from a new theory of gunnery, which he had invented, in order to verify the principles on which it was founded. The success was fully answerable to his expectations, and left no doubt of the truth of his theory, at least when applied to such pieces and bullets as he had used. These however were but small, being only musket balls of about an ounce weight: for, on account of the great size of the machinery necessary for such experiments, Mr. Robins, and other ingenious gentlemen, have not ventured to extend their practice beyond bullets of that kind, but contented themselves with ardently wishing for experiments to be made in a similar manner with balls of a larger size. By the experiments described in my paper therefore I endeavoured, in some degree, to

supply that defect, having used cannon balls of above twenty times the size, or from 1 pound to near 3 pounds weight. These are the only experiments, that I know of, which have been made in that way with cannon balls, though the conclusions to be deduced from such a course, are of the greatest importance, in those parts of natural philosophy which are connected with the effects of fired gunpowder: nor do I know of any other practical method besides that above, of ascertaining the initial velocities of military projectiles, within any tolerable degree of the truth; except that of the recoil of the gun, hung on an axis in the same manner as the pendulum; which was also first pointed out and used by Mr. Robins, and which has lately been practised also by Benjamin Thompson, Esq. (now Count Rumford) in his very ingenious set of experiments with musket balls, described in his paper in the Philosophical Transactions for the year 1781. The knowledge of this velocity is of the utmost consequence in gunnery: by means of it, together with the law of the resistance of the medium, every thing is determinable which relates to that business; for, as remarked in the paper above-mentioned on the first experiments, it gives us the law relative to the different quantities of powder, to the different weights of balls, and to the different lengths and sizes of guns; and it is also an excellent method of trying the strength of different sorts of powder. Besides these, there does not seem to be any thing wanting to answer every inquiry that can be made concerning the flight and ranges of shot, except the effects arising from the resistance of the medium.

2. In that course of experiments were compared, the effects of different quantities of powder, from 2 to 8 ounces; the effects of different weights of shot; and the effects of different sizes of ball, or various degrees of windage, being the difference between the diameter of the shot and the diameter of the bore: all of which were found to observe certain regular and constant laws, as far as the experiments were carried. And at the end of each day's

experiments, the deductions and conclusions were made, and the reasons clearly pointed out, why some cases of velocity differ from others, as they properly and regularly ought to do. So that it is surprizing how they could be misunderstood by Mr. Templehof, captain in the Prussian artillery, when speaking of the irregularities of such experiments, he says, (page 126 of *Le Bombardier Prussien*, printed at Berlin, 1781) "La meme chose arriva a Mr. Hutton, il la trouva de 626 pieds, & le jour suivant de 973 pieds, *tout les circonstances étant d'ailleurs égales*:" which last words show that Mr. T. had either misunderstood, or had not read the reason, which is a very sufficient one, for this remarkable difference: it is expressly remarked in page 171 of my paper in the Philosophical Transactions, that *all the circumstances* were *not* the same, but that the one ball was much smaller than the other, and that it had the less degree of velocity, 626 feet, because of the greater loss of the elastic fluid by the windage in the case of the smaller ball. On the contrary, the velocities in those experiments were even more uniform and similar than could be expected in such large machinery, and in a first attempt of the kind too. And from the whole, the following important conclusions were fairly drawn and stated.

"(1.) And first, it is made evident by these experiments, that powder fires almost *instantaneously*, seeing that nearly the whole of the charge fires, though the time be much diminished.

"(2.) The velocities communicated to shot of the same weight, with different quantities of powder, are nearly in the subduplicate ratio of those quantities. A very small variation, in defect, taking place when the quantities of powder become great.

"(3.) And when shot of different weights are fired with the same quantity of powder, the velocities communicated to them, are nearly in the reciprocal subduplicate ratio of their weights.

"(4.) So that, universally, shot which are of different weights, and impelled by the firing of different quantities of powder, acquire velocities which are directly as the square roots of the quantities of powder, and inversely as the square roots of the weights of the shot, nearly.

"(5.) It would therefore be a great improvement in artillery, to make use of shot of a long form, or of heavier matter; for thus the momentum of a shot, when fired with the same weight of powder, would be increased in the ratio of the square root of the weight of the shot.

"(6.) It would also be an improvement, to diminish the windage; for, by so doing, one third or more of the quantity of powder might be saved.

"(7.) When the improvements mentioned in the last two articles are considered as both taking place, it is evident that about half the quantity of powder might be saved; which is a very considerable object. But, important as this saving may be, it seems to be still exceeded by that of the guns: for thus a small gun may be made to have the effect and execution of one of two or three times its size, in the present way, by discharging a long shot of two or three times the weight of its natural ball, or round shot: and thus a small ship might discharge shot as heavy as those of the greatest now made use of.

"Finally, as the above experiments exhibit the regulations with regard to the weight of powder and balls, when fired from the same piece of ordnance; so, by making similar experiments with a gun, varied in its length, by cutting off from it a certain part before each course of experiments, the effects and general rules for the different lengths of guns, may with certainty be determined by them. In short, the principles on which these experiments were made, are so fruitful in consequences, that, in conjunction with the effects of the resistance of the medium, they appear to be sufficient for answering all the inquiries of the speculative philosopher, as well as those of the practical artillerist."

3. Such then was the state of the first set of experiments with cannon balls, made in the year 1775: and such were the probable advantages to be derived from them. I do not however know that any use has hitherto been made of them by authority for the public service; unless perhaps we are to except the instance of Carronades, a species of ordnance which hath since been invented, and in some degree adopted in the public service; for, in this instance, the proprietors of those pieces, by availing themselves of the circumstances of large balls, and very small windage, with small charges of powder, have been able to produce very considerable and useful effects with those light pieces, at a comparatively small expence. Or perhaps those experiments were too much limited, and of too private a nature, to merit a more general notice. Be that however as it may, the present additional course, which is to make the subject of this tract, will have great advantages over the former, both in point of extent, variety, improvements in machinery, and in authority. His Grace the Duke of Richmond, the master-general of the ordnance, in his indefatigable endeavours for the good of the public service, was pleased to order this extensive course of experiments, and to give directions for providing guns, and machinery, and every thing complete and fitting for the proper execution of them.

4. This course of experiments has been carried on under the direction of Major Bloomfield, (now General Sir Thomas B,) inspector of artillery, an officer of great professional merit, and whose ingenious contrivances in the machinery do him the utmost credit. It has been our employment for several successive summers, namely, those of the years 1783, 1784, 1785, 1787, 1788, 1789, 1791, &c; and indeed it might be continued much longer, either by extending it to more objects, or to more repetitions of experiments for the same object.

5. The objects of this course have been various. But the principal articles of it are as follow:

(1.) The velocities with which balls are projected by

equal charges of powder, from pieces of the same weight and calibre, but of different lengths.

(2.) The velocities with different charges of powder, the weight and length of the gun being the same.

(3.) The greatest velocity due to the different lengths of guns; to be obtained by increasing the charge as far as the resistance of the piece is capable of sustaining.

(4.) The effect of varying the weight of the piece; every thing else being the same.

(5.) The penetration of balls into blocks of wood.

(6.) The ranges and times of flight of balls, with the velocities by striking the pendulum at various distances; to compare them with their initial velocities, for determining the resistance of the medium.

(7.) The effect of wads;

of different degrees of ramming, or compressing the charge;

of different degrees of windage;

of different positions of the vent;

of chambers, and trunnions, and every other circumstance necessary to be known for the improvement of artillery.

On the Nature of the Experiment, and on the Machinery used in it.

6. The effects of most of the circumstances last mentioned, are determined by the actual velocity, with which the ball is projected from the mouth of the piece. Therefore the primary object of the experiments is, to discover that velocity in all cases, and especially in such as usually occur in the common practice of artillery. This velocity is very great; from 1000 to 2000 feet, or more, in a second of time. For conveniently estimating so great a velocity, the first thing necessary is, to reduce it, in some known proportion, to a small one. Which we may conceive to be effected in this manner: suppose the ball, projected with a great velocity, to strike some very heavy body, such as a large block of wood, from which it will not rebound, so that,

after the stroke, they may both proceed forward together with a common velocity. By this means, it is obvious that the original velocity of the ball may be reduced in any proportion, or to any slow motion which may conveniently be measured, by making the body struck to be sufficiently large: for it is well known that the common velocity, with which the ball and the block of wood would move on together, after the stroke, bears to the original velocity of the ball before the stroke, the same ratio which the weight of the ball has to that of the ball and block together. Thus then the velocities of 1000 feet in a second, are easily reduced to those of 2 or 3 feet only; which small velocity being measured by any convenient means, and the number denoting it increased in the ratio of the weight of the ball, to the weight of the ball and block together, the original velocity of the ball itself will thereby be obtained.

7. Now this reduced velocity is rendered easy to be measured by a very simple and curious contrivance, of Mr. Robins, which is this: the block of wood, which is struck by the ball, instead of being left at liberty to move straight forward in the direction of the ball's motion, is suspended, like the weight of the vibrating pendulum of a clock, by a strong iron stem, having a horizontal axis at the top, on the ends of which it vibrates freely when struck by the ball. The consequence of this simple contrivance is evident: this large ballistic pendulum, after being struck by the ball, is penetrated by it to a small depth, and it then swings round its axis, describing an arch, which is greater or less according to the force of the blow struck; and from the magnitude of the arch described by the vibrating pendulum, the velocity of any point of the pendulum can be easily computed: for, a body acquiring the same velocity by falling from the same height, whether it descend perpendicularly down, or otherwise: therefore, having given the length of the arc described by the centre of oscillation, or any other point, and its radius, the versed sine becomes known, which is the height perpendicularly descended by

that point of the pendulum. The height descended being thus known, the velocity acquired in falling through that height becomes known also, from the common rules for the descent of bodies by the force of gravity. And the velocity of this centre, thus obtained, is to be esteemed the velocity of the whole pendulum itself; which being now given, that of the ball before the stroke becomes known, from the given weights of the ball and pendulum. Thus then, the determination of the very great velocity of the ball, is reduced to the simple mensuration of the magnitude of the arch described by the pendulum, in consequence of the blow struck.

8. Now this arch may be determined in various ways: in the following experiments it was ascertained by measuring the length of its chord, which is the most useful line about it for making the calculations by; and this chord was measured sometimes by means of a piece of tape or narrow ribbon, the one end of which was fastened to the bottom of the pendulum, and the rest of it made to slide through a small machine contrived for the purpose; and sometimes it was measured by the trace of the fine point of a stylette in the bottom of the pendulum, made in an arch just below it, concentric with the axis, and covered with a composition of a proper consistence; which will be particularly described hereafter.

9. Another similar method of measuring the great velocity of the ball is, by observing the arch of recoil of the gun, when it is hung also after the manner of a pendulum: for, by loading the gun with adventitious weight, it may be made so heavy, as to swing any convenient extent of arch we please; which arch it is evident will be greater or less according to the velocity of the ball, or force of the inflamed powder, since action and reaction are equal and contrary; that is, the velocity of the ball will be greater than the velocity of the centre of oscillation of the gun, in the same proportion as the weight of the gun exceeds the weight of the ball. And therefore, if the velocity of the centre of oscillation of the gun be computed, from the chord of the

arc described by it in the recoil, the velocity of the ball will be found by this proportion; namely, as the weight of the ball is to the weight of the gun, so is the velocity of the gun, to the velocity of the ball: that is, if the weight of powder had no effect on the recoil.

10. This description may suffice to convey a general idea of the nature and principles of the experiment, for determining the velocity with which a ball is projected, by any charge of powder, from a piece of ordnance. But it is to be observed that, besides the centre of oscillation, and the weights of the ball and pendulum, or gun, the effect of the blow depends also on the place of the centre of gravity in the pendulum or gun, and that of the point struck, or the place where the force is exerted; for it is evident that the arch of vibration will be greater or less, according to the situation of these two points also. It will therefore be necessary now to give a more particular description of the machinery, and of the methods of finding the aforesaid requisites; and then we shall investigate our general rules for determining the velocity of the ball, in all cases, from them and the chord of the arc of vibration, either of the pendulum or gun.

Of the Guns, Powder, Balls, and Machinery employed in these Experiments.

11. Five very fine brass one-pounder guns were cast and prepared, in Woolwich Warren, for these experiments, and bored as true as possible; the common diameter of their bore being 2·02, or 2 inches and $\frac{2}{100}$ parts of an inch. Three of these, namely, nos. 1, 2, 3, were nearly of the same weight, but of the respective lengths of 15, 20, and 30 calibers; in order to ascertain the effect of different lengths of bore, with the same weight of gun, powder, and ball. The other two, nos. 4 and 5, were heavier, and of 40 calibers in length; in order to obtain the effects of the longest pieces.

N°. 5 was more expressly to show the effect of different lengths of the same gun: and for this purpose, it was to be fired a sufficient number of rounds with its whole length; and then to be successively diminished, by sawing off it 6 or 12 inches at a time, till it should be all cut away: firing a number of rounds with it at each length. And for the convenience of suspending this gun near its centre of gravity, for all the different lengths of it, a long thin slip was cast with it, extending along the under side of it, from the breech to almost the middle of its length. By perforating this slip through with holes immediately under the centre of gravity for each length, after being cut, a bolt was to pass through the hole, on which the gun might be suspended. The other guns were slung by their trunnions. The exact weight and dimensions of all these guns are exhibited in the following table.

No. of the gun	Length of the Piece, in calib.	Length of the Piece, in inch	Length of the Bore, in calib.	Length of the Bore, in inch	Diameter at the breech inch	Diameter at the muzzle inch	Diam. of the bore inch	Weight lb.
1	15	30·3	14·12	28·53	7·85	6·88	2·02	290
2	19·98	40·35	19·02	38·43	7·43	5·92	2·02	289
3	29·2	60	28·56	57·70	6·73	4·68	2·02	295
4	41·04	82·9	39·70	80·28	6·1	4·31	2·02	378
5	40·84	82·5	40·00	80·80	6·47	4	2·02	502

12. As these guns were to be slung by their trunnions, to observe the relation between the velocity of the ball and the arch of recoil described by the gun, vibrating on an axis, certain leaden weights were cast, to fit on very exactly about the trunnions of the gun, to render it so heavy, as that the arch of recoil might not be inconveniently great. These consisted, first of central pieces to fit the trunnions, and then over them cylindrical rings of different sizes, both turned to fit exactly; the whole being held firmly together by iron bolts put into holes bored through all the pieces. These were also of different sizes, so as to bring all the guns exactly up to the same weight; the whole weight of

each, together with 188 lb weight of iron, about the stem and machine, by which the gun was slung, was 917 lb; with which weight most of the experiments were made: notice being always taken when any alteration was made in the weights, as well as in the other circumstances. The common weight of 917 lb is made up of the different guns with leads, and the common weight of iron, as below:

No.	Guns		Leads		Iron		Total.
1 - - - - -	290	+	439	+	188	=	917
2 - - - - -	289	+	440	+	188	=	917
3 - - - - -	295	+	434	+	188	=	917
4 - - - - -	378	+	351	+	188	=	917
5 - - - - -	502	+	227	+	188	=	917

These were the weights at first; but soon after, the braces, or strengthening rods of the gun frame, were made longer and thicker, which added 11 lb to their weights, and then the whole weight of each was 928 lb.

13. In these experiments, the velocity of the ball, by which the force of the powder is determined, was to be measured both by the ballistic pendulum into which the ball was fired, and by the arch of recoil of the gun, which was hung on an axis by an iron stem, after the same manner as the pendulum itself, and the arcs vibrated in both cases measured in the same way. Plates V and VI contain general representations of the machinery of both; namely, a side view and a front view of each, as they hung by their stem and axis on the wooden supports. In plate V, fig. 1 is the side-view of the pendulum, and fig. 2 the side-view of the gun, as slung in their frames. And in plate VI, fig. 1 and 2 are the front-views of the same.

14. In fig. 1, of both plates, A is the pendulous block of wood, into which the balls are fired, strongly bound with thick bars of iron, and hung by a strong iron stem, which is connected by an axis at top; the whole being firmly braced together by crossing diagonal rods of iron. The cylindrical ends of the axis, both in the gun and pendulum,

were at first placed to turn upon smooth flat plate-iron surfaces, having perpendicular pins put in before and behind the sides of the axis, to keep it in its place, and prevent it from slipping backwards and forwards. But, this method being attended with too much friction, the ends of the axis were afterwards supported and made to roll upon curved pieces, having the convexity upwards, and the pins, before and behind the axis, set so as not quite to touch it; which left a small degree of play to the axis, and made the friction less than before. But, still further to diminish the friction, the lower side of the ends of the axis was narrowed off a little, something like the axis of a scale beam, and made to turn in hollow grooves, which were rounded down at both ends, and standing higher in the middle, like the curvature of a bent cylinder; by which means the edge of the axis touched the grooves, not in a line, but in one point only; and then it vibrated with very great freedom, having only an almost imperceptible degree of friction. The several times and occasions when these, and other improvements, were introduced and used, are more particularly noticed in the journal of the experiments.

15. At first, the chord of the arc, of vibration and recoil, was measured by means of a prepared narrow tape, divided into inches and tenths, as before. A new contrivance of machinery was however made for it. From the bottom of the pendulum, or gun-frame, proceeded a tongue of iron, which was raised or lowered by means of a screw at B; this was cloven at the bottom C, to receive the end of the tape, and the lips then pinched together by a screw, which held the tape fast. Immediately below this the tape was passed between two slips of iron, which could be brought to any degree of approach by two screws; these pieces were made to slide vertically up and down a groove in a heavy block of wood, and fixed at any height by a screw D. One of these latter pieces was extended out a considerable length, to prevent the tape from getting over its ends, and entangling in the returns of the vibrations. The extent of

tape drawn out in a vibration, it is evident, is the chord of the arc described, and counted in inches and tenths, to the radius measured from the middle of the axis to the bottom of the tongue.

16. This method however was still found to be attended with some trouble, and many inconveniences, as well as doubts and uncertainty sometimes. For which reasons we afterwards changed this way of measuring the chord of vibration, for another, which answered much better in every respect. This consisted in a block of wood, having its upper surface EF formed into a circular arc, whose centre was in the middle of the axis, and consequently its radius equal to the length from the axis to the upper surface of the block. In the middle of this arch was made a shallow groove of 3 or 4 inches broad, running along the middle, through the whole length of the arch. This groove was filled with a composition of soft-soap and wax, of about the consistence of honey, or a little firmer, the upper side being smoothed off, even with the general surface of the broad arch. A sharp spear or stylette then proceeded from the bottom of the pendulum or gun-frame, and so low as just to enter and scratch along the surface of the composition in the groove, without having any sensible effect in retarding the motion of the body. The trace remaining, the extent of it could easily be measured. This measurement was effected in the following manner:—A line of chords was laid down upon the upper surface of the wooden arch, on each side of the groove, and the divisions marked with lines on a ground of white paint: the edge of a straight ruler being then laid across by the corresponding divisions, just to touch the farthest extent of the trace in the composition, gave the length of the chord as marked on the arch. To make the computations by the rule for the velocity easier, the divisions on the chords were made exact thousandth parts of the radius, which saved the trouble of dividing by the radius at every operation. The manner in which this line of chords was constructed on the face

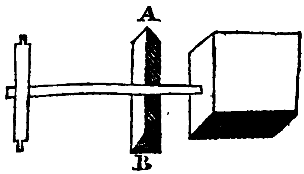
of the arch, was this: The radius was made just 10 feet: I therefore prepared a smooth and straight deal rod, upon which was set off a length of 10 feet; I then divided each foot into 10 equal parts, and each of these into 10 parts again; by which means the whole rod or radius was divided into 1000 equal parts, being 100th parts of a foot. I then transferred the divisions of the rod to the face of the arch in this manner, namely; the first division of the rod was applied to the side of the arch at the beginning of it, and made to turn round there as a centre; then, in that position, the rod, when turned vertically round that point, always touched the side of the arch, and the divisions of it were marked on the edge of the arch, successively as they came into coincidence with it.

17. In fig. 2, plate V, G shows the leaden weights placed about the trunnions; H a screw for raising or depressing the breech of the gun, by means of the piece I embracing the cascable, and moveable along the perpendicular arm KL, to suit the different lengths of guns, and held to it by a screw passing through the slit made along it. The machines and operations for finding the ranges will be described hereafter.

On the Centres of Gravity aud Oscillation.

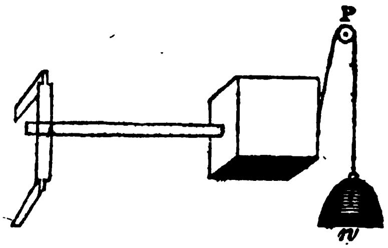
18. It being necessary to know the position of the centres of gravity and oscillation, without which, the velocity cannot be computed; these were commonly determined every day as follows:

The centre of gravity was found by one or both of these two methods. First, a triangular prism of iron A B, being placed on the ground with one edge upwards, the pendulum or gun-frame was laid across it, and moved backward or forward, on the stem or block, as the case required, till the two parts exactly balanced each



other in a horizontal position. Then, as it lay, the distance was measured from the middle of the axis to the part which rested on the edge of the prism, or the place of the centre of gravity, which is the distance g of that centre below the axis.

19. The other method is this: The ends of the axis being supported on fixed uprights, and a chord fastened to the lower end of the block, or of the gun-frame, and passed over a pulley at P, different weights w were fastened to the other end of it, till the body was brought to a horizontal position. Then, taking also the whole weight of the body, and its length from the axis to the bottom, where the chord was fixed, the place of the centre of gravity is found by this proportion:



As p the weight of the pendulum :
is to w the appended weight : :
so is d the whole length from the axis to the chord :
to $\frac{dw}{p}$ the distance from the axis to the centre of gravity.

Either of these two methods gave the place of the centre of gravity sufficiently exact; but the agreement of the results of both of them was still more satisfactory.

20. To find the centre of oscillation, the ballistic pendulum, or the gun, was hung up by its axis in its place, and then made to vibrate in small arcs, for 1 minute, or 2, or 5, or 10 minutes; the more the better; as determined either by a half seconds pendulum, or a stop watch, or a peculiar time-piece, measuring the time to 40th parts of a second; and the number of vibrations performed in that time carefully counted. Having thus obtained the time answering to a certain number of vibrations, the centre of oscillation is easily found: for, if n denote the number of

vibrations made in s seconds, and l the length of the seconds pendulum, then it is well known that $n^2 : s^2 :: l : \frac{s^2 l}{n^2}$ the distance from the axis of motion to the centre of oscillation. And here if s be 60 seconds, or one minute, and n the number of vibrations performed in 1 minute, as found by dividing the whole number of vibrations, actually performed, by the whole number of minutes; then is $n^2 : 60^2 :: l : \frac{3600l}{nn}$ the distance to the centre of oscillation. But, by the best observations on the vibration of pendulums, it is found that $l = 39\frac{1}{8}$ inches, is the length of the seconds pendulum for the latitude of London, or of Woolwich; and therefore $\frac{3600 \times 39\frac{1}{8}}{nn}$ or $\frac{140850}{nn} = o$, will be the distance, in inches, or $= \frac{11737 \cdot 5}{nn}$ in feet, of the centre of oscillation below the axis. And by this rule the place of that centre was found for each day of the experiments.

Of the Rule for Computing the Velocity of the Ball.

21. Having described the methods of obtaining the necessary dimensions and weights, proceed we now to the investigation of the theorem by which the velocity of the ball is to be computed: and first by means of the pendulum.

The several weights and measures being found, let

b denote the weight of the ball,
p the weight of the pendulum,
g the distance to its centre of gravity,
o the distance to its centre of oscillation,
i the distance to the point of impact, or point struck,
c the chord of the arch described by the pendulum,
r its radius, or distance to the tape or arch,
v the initial or original velocity of the ball, at the impact.

Then, from the nature of oscillatory motion, bii will express the sum of the forces of the ball acting at the distance i from the axis, and pgo the sum of the forces of the pendulum, and consequently $pgo + bii$ the sum for

both the ball and pendulum together; and if each be multiplied by its velocity, $biiv$ will be the quantity of motion of the ball, and $(pgo + bii) \times z$ the quantity for the pendulum and ball together; where z is the velocity of the point of impact. But these quantities of motion, before and after the blow, must be equal to each other, therefore $(pgo + bii) \times z = biiv$, and consequently $z = \frac{biiv}{pgo + bii}$ is the velocity of the point of impact. Now, because of the accession of the ball to the pendulum, the place of the centre of oscillation will be changed; and the distance y, of the new or compound centre of oscillation, will be found by dividing $pgo + bii$ the sum of the forces, by $pg + bi$ the sum of the momenta, that is, $y = \frac{pgo + bii}{pg + bi}$ is the distance of the new or compound centre of oscillation below the axis. Then, because $\frac{biiv}{pgo + bii}$ is the velocity of the point whose distance is i, by similar figures we shall have this proportion, as $i : \frac{pgo + bii}{pg + bi}$ (or y) $:: \frac{biiv}{pgo + bii} : \frac{biv}{pg + bi}$, the velocity of this compound centre of oscillation.

Again, by the property of the circle, $2r : c :: c : \frac{cc}{2r}$, which will be the versed sine of the described arc, to the chord c and radius r; and hence, by similar figures, $r : y$ or $\frac{pgo + bii}{pg + bi} :: \frac{cc}{2r} : \frac{cc}{2rr} \times \frac{pgo + bii}{pg + bi}$ the corresponding versed sine to the radius y, or the versed sine of the arc described by the compound centre of oscillation; which call v. Then, because the velocity lost in ascending through the circular arc, or gained in descending through the same, is equal to the velocity acquired in descending freely by gravity through its versed sine, or perpendicular height, therefore the velocity of this centre of oscillation will also be equal to the velocity generated by gravity in descending through the space v or $\frac{cc}{2rr} \times \frac{pgo + bii}{pg + bi}$. But the space described by gravity in one second of time, in the latitude of London, is 16·09 feet, and the velocity generated in that time 32·18; therefore, by the nature of free descents, $\sqrt{16 \cdot 09} : \sqrt{\text{v}} :: 32 \cdot 18 : \frac{5 \cdot 6727\, c}{r} \sqrt{\frac{pgo + bii}{pg + bi}}$, the velocity of the same centre of

oscillation, as deduced from the chord of the arc which is actually described.

Having thus obtained two different expressions for the velocity of this centre, independent of each other, let an equation be made of them, and it will express the relation of the several quantities in the question: thus then we have $\frac{biv}{pg+bi} = \frac{5{\cdot}6727\,c}{r}\sqrt{\frac{pgo+bii}{pg+bi}}$. And from this equation we obtain $v = \frac{5{\cdot}6727\,c}{bir}\sqrt{[(pgo+bii)\times(pg+bi)]}$, the true expression for the original velocity of the ball the moment before it strikes the pendulum. And this theorem agrees with those of Messrs. Euler and Antoni, and also with that of Mr. Robins nearly, for the same purpose, when his rule is corrected by the parapraph which was by mistake omitted in his book when first published; which correction he himself gave in a paper in the Philosophical Transactions for April 1743, and where he informs us that all the velocities of the balls, mentioned in his book, except the first only, were computed by the corrected rule. Though the editor of his works, published in 1761, has inadvertently neglected this correction, and printed his book without taking any notice of it. And that remark, had M. Euler observed it, might have saved him the trouble of many of his animadversions on Mr. Robins's work.

22. But this theorem may be reduced to a form much more simple and fit for use, and yet be sufficiently near the truth. Thus, let the root of the compound factor $(pgo+bii)\times(pg+bi)$ be extracted, and it will be equal to $(pg+bi\,.\,\frac{o+i}{2o})\times\sqrt{o}$, within the 100000th part of the true value, in such cases as commonly occur in practice. But, since $bi\,.\,\frac{o+i}{2o}$, in our experiments, is usually but about the 500th, or 600th, or 800th part of pg; and since bi differs from $bi\,.\,\frac{o+i}{2o}$ only by about the 100th part of itself; therefore $pg+bi$ is within the 50000th part of $pg+bi\,.\,\frac{o+i}{2o}$. Consequently $v = 5{\cdot}6727\,c\,.\,\frac{pg+bi}{bir}\sqrt{o}$ very nearly. Or, further, if g be written for i in the last term bi, then finally $v =$

$5{\cdot}6727\,gc.\frac{p+b}{bir}\sqrt{o}$; which is an easy theorem to be used on all occasions; and being within the 5000th part of the true quantity, it will always give the velocity true within less than half a foot, even in the cases of the greatest velocities. Where it must be observed, that c, g, i, r, may be taken in any measures, either feet or inches, &c, provided they be but all of the same kind; but o must be in feet, because the theorem is adapted to feet.

23. As the balls remain in the pendulum during the time of making one whole set of experiments, both its weight, and the position of the centres of gravity and oscillation, will be changed by the addition of each ball which is lodged in the wood; and therefore p, g, o must be corrected after every shot, in the theorem for determining the velocity v. Now the succeeding value of p is always $p+b$; or p is to be corrected by the continual addition of b: and the succeeding value of g is $g+\frac{i-g}{p+b}b$, or $g+\frac{i-g}{p}b$, nearly; or g is corrected by adding always $\frac{i-g}{p}b$ to the next preceding value of g: and lastly, o is to be corrected by taking for its new values successively $\frac{pgo+bii}{pg+bi}$, or by adding always $\frac{i-o}{pg+bi}bi$, or $\frac{i-o}{p}b$ nearly, to the preceding value of o: so that the three corrections are made by adding always,

b to the value of p,

$\frac{i-g}{p}\times b$ to the value of g,

$\frac{i-o}{p}\times b$ to the value of o.

That is, when b is very small in respect of p.

24. But as the distance of the centre of oscillation o, whose square root is concerned in the theorem for the velocity v, is found from the number of vibrations n performed by the pendulum; it will be better to substitute, in that theorem, the value of o in terms of n. Now by Art. 20, the value of o is $\frac{11737{\cdot}5}{nn}$ feet, and consequently $\sqrt{o}=\frac{108{\cdot}3398}{n}$; which value of $\sqrt{o}$ being substituted for it, in the theorem $v=5{\cdot}6727\,gc\times\frac{p+b}{bir}\sqrt{o}$, it becomes $v=614{\cdot}58\,gc\times\frac{p+b}{birn}$, or

$=\frac{59000}{96}\times\frac{p+b}{birn}gc$, the simplest and easiest formula for the velocity of the ball in feet: where c, g, i, r may be taken in any one and the same measure, either all inches, or all feet, or any other measure.

25. It will be necessary here to add a correction for n, instead of that for o in Art. 23. Now, the correction for o being $\frac{i-o}{pg+bi}b\,i$, and the value of $n=\frac{375\cdot3}{\sqrt{o}}$ inches, the correction for n will be $375\cdot3\times$

$$\left(\frac{1}{\sqrt{o}}-\frac{1}{\sqrt{(o+\frac{i-o}{pg+bi}\cdot bi)}}\right)=375\cdot3\times\left(\frac{1}{\sqrt{o}}-\frac{1}{\sqrt{o}\times\sqrt{(1+\frac{i-o}{pg+bi}\cdot\frac{bi}{o})}}\right)$$

$$=n-\frac{n}{\sqrt{(1+\frac{i-o}{pg+bi}\cdot\frac{bi}{o})}}=n-\frac{n}{1+\frac{i-o}{pg+bi}\cdot\frac{bi}{2o}}\text{ nearly }=$$

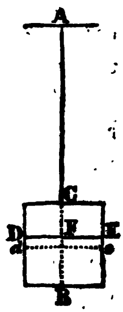
$\frac{bin\times(inn-140850)}{281700pg+bi\times(inn+140850)}$ by substituting the value of o instead of it: Which correction is negative, or to be subtracted from the former value of n. The corrections for p and g being b and $\frac{i-g}{p+b}b$, as in Art. 23; which are both additive. But the signs of these quantities must be changed when b is negative.

26. Before quitting this rule, it may be necessary here to advert to three or four circumstances which may seem to cause some small error in the initial velocity, as determined by the formula in Art. 24. These are the friction on the axis, the resistance of the air to the back of the pendulum, the time which the ball employs in penetrating the wood of the pendulum, and the resistance of the air to the ball in its passage between the gun and the pendulum.

As to the first of these, namely, the friction on the axis, by which the extent of its vibration is somewhat diminished; it may be observed, that the effect of this cause can never amount to a quantity considerable enough to be brought into account in our experiments: for, besides that care was taken to render this friction as small as possible, the effect of the small part which does remain, is nearly balanced by the effect it has on the number n of vibrations performed in a minute; for the friction on the axis will

a little retard its motion, and cause its vibrations to be slower, and fewer; so that c the length of a vibration, and n the number of vibrations, being both diminished by this cause, nearly in an equal degree, and c being a multiplier, and n a divisor, in our formula, it is evident that the effect of the friction in the one case, operates against that in the other, and that the difference of the two is the real disturbing cause, and which therefore is either equal to nothing, or very nearly so.

27. The second cause of error, is the resistance of the air against the back of the pendulum, by which its motion is somewhat impeded. This resistance hinders the pendulum from vibrating so far, and describing so large an arch, as it would do, if there was no such resistance; therefore the chord of the arc, which is actually described and measured, is less than it really ought to be; and consequently the velocity of the ball, which is proportional to that chord, will be less than the real velocity of the ball at the moment it strikes the pendulum. And though the pendulum be very heavy, and its motion but slow, and consequently the resistance of the air against it very small, it will yet be proper to investigate the real effect of it, that we may ascertain whether it may safely be neglected or not.



In order to this, let the annexed figure represent the back of the pendulum, moving on its axis; and put

p = weight of the pendulum,
a = DE its breadth,
r = AB the distance to the bottom,
e = AC the distance to the top,
x = AF any variable distance,
g = distance of the centre of gravity,
o = distance of the centre of oscillation,
v = velocity of the centre of oscillation, in any part of the vibration,
h = 16·09 feet, the descent of gravity in 1 second,
c = the chord of the arc actually described by the centre of oscillation,

c = the chord which would be described by it if the air made no resistance.

Then $o : x :: v : \frac{vx}{o}$ the velocity of the point F of the pendulum; and $4h^2 : \frac{v^2x^2}{o^2} :: h : \frac{v^2x^2}{4ho^2}$ the height descended by gravity to generate the velocity $\frac{vx}{o}$. Now the resistance of the air to the line D F E, is nearly equal to the pressure of a column of air upon it, whose height is the same $\frac{v^2x^2}{4ho^2}$, and therefore that pressure or weight is $\frac{nav^2x^2}{4ho^2}$, where n is the specific gravity, or weight of one cubic measure of air, or $n = \frac{62\frac{1}{2}}{850}$lb. $= \frac{5}{68}$lb. Hence then $\frac{nav^2x^2\dot{x}}{4ho^2}$ is the pressure on D E e d, and $\frac{nav^2x^3\dot{x}}{4ho^2}$ the motive force on the same D e, or the fluxion of the force on the block of the pendulum; and the correct fluent gives $na \times \frac{r^4 - e^4}{16ho^2}v^2$ for the force of the air on the whole pendulum, supposing that on the stem A C to be nothing, as it is nearly, both on account of its narrowness, and the diminution of the momentum of the particles by their nearness to the axis. Put now A = the compound coefficient $na \times \frac{r^4 - e^4}{16ho^2}$, so shall $\text{A}v^2$ denote the motive force of the air on the back of the pendulum.

But the motion of the pendulum is also obstructed by its own weight, as well as by the resistance of the air; and that weight acts as if it were all concentred in the centre of gravity, whose distance below the axis is g; therefore pg is its momentum in its natural or vertical direction, and pgs its momentum perpendicular to the motion of the pendulum, where s is the sine of the angle which it makes at any time with the vertical position, to the radius 1. Hence $pgs + \text{A}v^2$ is the momentum of both the resistances together, namely that of the pressure of the air, and of the weight of the pendulum. Consequently $\frac{pgs + \text{A}v^2}{pg} = s + \frac{\text{A}}{pg}v^2$ is the real retarding force to the motion of the pendulum, at the centre of oscillation; which force call f.

Now if z denote the arc described by the centre of oscillation, when its velocity is v, or $\frac{z}{o}$ the arc whose sine is s; we shall have

$\dot z = \frac{o\dot s}{\sqrt{(1-s^2)}}$, and, by the doctrine of forces,

$$v\dot v = -2hf\dot z = -2ho.\frac{s\dot s}{\sqrt{(1-ss)}} - \frac{2ho\text{A}}{pg}.\frac{v^2\dot s}{\sqrt{(1-ss)}}.$$

But $\frac{cc}{2o}$ is the versed sine or height of the whole arc whose chord is c, and $o \times (1 - \sqrt{(1-ss)}) = o - o\sqrt{(1-ss)}$ is the versed sine or height of the part whose sine is os; therefore $\frac{cc}{2o} - o + o\sqrt{(1-ss)}$ is their difference, or the height of the remaining part, and which is nearly equal to the height due to the velocity v; therefore $h : 4hh ::$ $\frac{cc}{2o} - o + o\sqrt{(1-ss)} : v^2 = 4h \times (\frac{cc}{2o} - o + o\sqrt{(1-ss)})$ nearly. Then, by substituting this for v^2 in the value of $v\dot v$, we have

$$v\dot v = -2ho.\frac{s\dot s}{\sqrt{(1-ss)}} - \frac{8hho\text{A}}{pg} \times (\frac{c^2-2o^2}{2o}.\frac{\dot s}{\sqrt{(1-ss)}} + o\dot s);$$

and the fluents give

$$v^2 = 4ho\sqrt{(1-ss)} + \frac{16hho\text{A}}{pg}(\frac{2o^2-c^2}{2o^2}z - os) + \text{Q};$$

where Q is a constant quantity by which the fluent is to be corrected. Now, substituting v^2 for v^2, and 0 for s, their corresponding values at the commencement of motion, the above fluents become

$\text{v}^2 = 4ho + \text{Q}$; from which the former subtracted, gives

$$\text{v}^2 - v^2 = 4ho - 4ho\sqrt{(1-ss)} - \frac{16hho\text{A}}{pg}(\frac{2o^2-c^2}{2o^2}z - os).$$

And when $v = 0$, or the pendulum is at the full extent of its ascent, then

$$\text{v}^2 = 4ho - 4ho\sqrt{(1-ss)} - \frac{16hho\text{A}}{pg}(\frac{2o^2-c^2}{2o^2}z - os),$$

at which point os is the sine of the whole arc whose chord is c, and consequently $s = \frac{c}{2o^2}\sqrt{(4o^2-c^2)}$.

But the value of s being commonly small in respect of $\frac{c}{o}$, these following values will be nearly true, namely,

$$\sqrt{(1-ss)} = 1 - \tfrac{1}{2}s^2 - \tfrac{1}{8}s^4 = 1 - \frac{4c^2o^2-c^4}{8o^4} - \frac{c^4}{8o^4} = 1 - \frac{c^2}{2o^2},$$

$$4ho - 4ho\sqrt{(1-ss)} = 4ho.\frac{c^2}{2o^2},$$

$$z = os + \tfrac{1}{6}os^3, \text{ and}$$

$$\frac{2o^2-c^2}{o^2 2}z - os = -\frac{c^2s}{2o} + \frac{2o^2-c^2}{12o}s^3;$$

which values, by substitution, give

$$v^2 = \frac{2hc^2}{o} + \frac{16h^2 o A}{pg}\left(\frac{c^2 s}{2o} - \frac{2o^2 - c^2}{12o} s^3\right).$$

But $\frac{c^2}{2o}$ is the versed sine or height to the chord c, and $v^2 = 4h \cdot \frac{c^2}{2o} = \frac{2hc^2}{o}$ the square of the velocity due to that height; therefore

$$\frac{2hc^2}{o} = \frac{2hc^2}{o} + \frac{16h^2 o A}{pg}\left(\frac{c^2 s}{2o} - \frac{2o^2 - c^2}{12o} s^3\right), \text{ and}$$

$$c^2 = c^2 + \frac{8hoA}{pg}\left(\frac{c^2 s}{2} - \frac{2o^2 - c^2}{12} s^3\right), \text{ or}$$

$$c^2 = c^2 + \frac{8hA}{pg}\left(\frac{c^3}{3} + \frac{c^5}{12o^2}\right), \text{ and}$$

$$c = c + \frac{4c^2 hA}{3pg} \text{ nearly; or, substituting for A,}$$

$$c = c + \frac{nac^2}{12pg} \cdot \frac{r^4 - e^4}{o^2} = c\left(1 + \frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}\right).$$

So that the chord of the arc which is actually described, is to that which would be described if the air had no resistance, as 1 is to $1 + \frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}$; and therefore $\frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}$ is the part of the chord, and consequently of the velocity, lost by means of the resistance of the air. And the proportion is the same for the chords described by the lowest point, or any other point, of the pendulum.

Now, to give an example, in numbers, of this effect of the resistance of the air; the ordinary mean values of the literal quantities are as here below:

namely,	therefore		
$p = 700$	nac	$=$	$\frac{25}{102}$
$a = 2$			
$r = 8\frac{1}{2}$	$12pg$	$=$	56000
$e = 6\frac{1}{2}$	r^4	$=$	5220
$g = 6\frac{2}{3}$	e^4	$=$	1785
$o = 7\frac{1}{3}$	$r^4 - e^4$	$=$	3435
$n = \frac{5}{68}$	o^2	$=$	$\frac{484}{9}$
$c = 1\frac{2}{3}$	$\frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}$	$=$	$\frac{1}{3577}$

So that the part of the chord, or velocity, lost by this cause, namely, the resistance of the air on the back of the pendulum, is but about the $\frac{1}{3577}$, or about the $\frac{1}{4000}$ part of the

whole; and therefore this effect scarcely ever amounts to so much as half a foot: being indeed about

$\frac{1}{2}$ of a foot	when the velocity of the ball is				2000 feet,
$\frac{1}{4}$	-	when	- -	it is	1000
$\frac{3}{8}$	-	when	- -	it is	1500

and so on in proportion to the whole velocity of the ball.

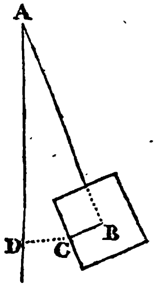
And even this small effect may be supposed to be balanced by the method of determining the centre of oscillation, or the number of vibrations made in a second. So that the number of oscillations, and the chord of the arc described, being both diminished by the resistance of the air; and the one of these quantities being a multiplier, and the other a divisor, in the formula for the velocity; the one of those small effects will nearly balance the other; much in the same way as the effects of the first cause, or the friction on the axis. So that these effects may both of them be safely neglected, as in no case amounting to any sensible quantity.

In the beginning of this investigation, it is supposed that the resistance of the medium is equal to the weight of a column of it, whose base is the moving surface, and its altitude equal to that from which a heavy body must fall to acquire the velocity of that surface. But some philosophers think the altitude should be only one half of that, and consequently the pressure only one half: which would render the resistance still less considerable. But if the altitude and resistance were even double of that above found, it might still be safely neglected.

28. The third seeming cause of error in our rule, is the time in which the ball communicates its motion to the pendulum, or the time employed in the penetration. The principle on which the rule is founded, supposes the momentum of the ball to be communicated in an instant: but this is not accurately the case, because this force is communicated during the time in which the ball makes the penetration. And though that time be evidently very small, scarcely amounting to the 500th part of a second, it will be proper to enquire what effect that circumstance may

have on the truth of our theorem, or on the velocity of the ball, as computed from it.

In order to this, let the notation employed in Art. 21 be supposed here; and let A B C be a side-view of the pendulum, moved out of the vertical position A D, by the perpendicular blow of the ball against the point D or C. Also



let x = D C the space moved by the point of impact C,

z = C B the depth penetrated by the ball,

v = velocity of the ball at B,

u = velocity of the point C of the pendulum, and

R = the uniform resisting force of the wood.

Then is $\frac{R}{b}$ the retarding force of the ball, which is constant. Again, as the motion of the pendulum arises from the resisting force R of the wood, Ri will be its momentum; and as the sum of the forces in the pendulum was found to be $= pgo$, the accelerating force of the point C will be $\frac{Rii}{pgo}$, which force is constant also. But, in the action of forces that are constant, the time t is equal to the velocity divided by the force, and by $2h$ or $2 \times 16{\cdot}09$ feet, and the space is equal to the square of the velocity divided by the force and by $4h$; consequently

$$t = \frac{pgou}{2hiiR}, \qquad x = \frac{pgouu}{4hiiR},$$

$$t = \frac{-b\mathrm{v}}{2hR}, \qquad x + z = \frac{-b\mathrm{vv}}{4hR},$$

or by correc. $t = \frac{b}{2hR} \times (v - \mathrm{v})$, $x + z = \frac{b}{4hR} \times (v^2 - \mathrm{v}^2)$.

The two values of the time t being equated, we obtain

$$pgou = bii(v - \mathrm{v}), \text{ or } pgou + bii\mathrm{v} = biiv.$$

And when $\mathrm{v} = u$, or the action of the ball on the pendulum ceases, this equation becomes $pgou = biiu + biiv$, and hence $u = \frac{biiv}{pgo + bii}$ is the greatest velocity of the point C, at the instant when the ball has penetrated to the greatest depth, and ceases to urge the pendulum farther. So that this velocity is the same, whatever the resisting force of the

wood is, and therefore to whatever depth the ball penetrates, and the same as if the wood were perfectly hard, or the ball made no penetration at all. And this velocity of the point of impact also agrees with that which was found in Art. 21. So that the velocity communicated to the point of impact is the same, whether the impulse is made in an instant, or in some small portion of time. And hence, in the usual case of a penetration, because the block will have moved some small distance before it has attained its greatest velocity, it might at first view seem as if it would swing or rise higher than when that velocity is communicated in an instant, or when the pendulum is yet in its vertical position, and so might describe a longer chord, and show a greater velocity of the ball than it ought. But, on the other hand, it must be considered, that in the small part of its swing, which the pendulum has made before the penetration is completed, or has attained its greatest velocity, just as much velocity will be lost by the opposing gravity or weight of the pendulum, as if it had set out from the vertical position with the said greatest velocity; and therefore the real velocity at that height will be the same in both cases. Hence then it may safely be concluded, that the circumstance of the ball's penetration causes no alteration in the velocity of it, as computed by our formula. And as it was before found that no sensible error is incurred by the two first circumstances, namely, the friction on the axis, and the resistance of the air to the back of the pendulum, we may be well assured that our formula brings out the true velocity with which the ball strikes the pendulum, without any sensible error.

29. Since $\frac{biiv}{pgo + bii}$ denotes the greatest velocity which the point C of the pendulum acquires by the stroke, dividing by i, we shall have $\frac{biv}{pgo + bii}$ for the angular velocity of the pendulum, or that of a radius 1. From which it appears, that the vibration will be very small when i or the distance AD is small, and also when i is very great. And if

we take this expression a *maximum*, and make its fluxion $= 0$, i only being variable, we shall obtain $pgo = bii$, and $i = \sqrt{\frac{pgo}{b}}$ for the distance of the centre of percussion, or the point where the ball must strike, so as to cause the greatest vibration in the pendulum; which point, in this case, is neither the centre of gravity nor the centre of oscillation; but will be at a great distance below the axis when p is great in respect of b, as in the case of our experiments, in which p is 600 or 800 times as much as b.

30. It may not be improper here, by the way, to enquire a little into the time of the penetration, its extent or depth, and the measure of the resisting force of the wood. It was found above, Art. 28, that

$$x = \frac{pgouu}{4hiiR}, \text{ and } x + z = \frac{b}{4hR} \times (v^2 - \text{v}^2).$$

Now, substituting in these, $\frac{biiv}{pgo + bii}$, the greatest value of u, for u and v, we have

$$\text{x} = \frac{pgo}{4hR} \times \frac{b^2 i^2 v^2}{(pgo + bii)^2}, \text{ and z} = \frac{pgo}{4hR} \times \frac{bv^2}{pgo + bii}.$$

The latter of these being the greatest depth penetrated by the ball into the wood, and the former the distance moved by the point c of the pendulum at the instant when the penetration is completed. Both of which, it is evident, are directly as the square of the original velocity of the ball, and inversely as the resisting force of the wood; the other quantities remaining constant.

Hence also it appears that, other things remaining, the penetration will be less, as i is greater, or as the point of impact is farther below the axis. It is further evident, that the penetration will diminish as the sum of the forces pgo diminishes.

Now, for an example in numbers, a ball fired with a velocity of 1500 feet per second, has been found to penetrate about 14 inches into a block of sound dry elm, when the dimensions of the pendulum were as below:

$p = 660$lb the ball being cast iron,
$g = 78$ inches or $6\frac{1}{2}$ feet, its diameter 1·96 inches,
$o = 84$ inches or 7 feet, and its weight $1\frac{3}{64}$ or $\frac{67}{64}$lb.
$i = 90$ inches or $7\frac{1}{2}$ feet, also the value of z is 14 inch. or $\frac{7}{6}$ feet.

Here the value of v is 1500, and $z = 14$ inches or $\frac{7}{6}$ feet.

Hence $R = \frac{pgo}{4hz} \times \frac{bv^2}{pgo + bii} = \frac{660.13.7.6.67}{2.4.16.7.64} \times \frac{1500.1500}{660.6\frac{1}{2}.7 + \frac{67}{64}.\frac{15}{2}.\frac{15}{2}}$

$= 32000$ nearly, which is the value of R for a ball of that size and weight. Or the resistance in this instance is 32000 times the force of gravity.

Hence also $x = \frac{pgo}{4hR} \times \frac{b^2 i^2 v^2}{(pgo + bii)^2} = \frac{1}{469}$ part of a foot, or $\frac{1}{39}$ part of an inch, is the space moved by the point C of the pendulum when the penetration is completed.

Also $T = \frac{pgou}{2hiiR} = \frac{2z}{v} = \frac{\frac{7}{3}}{1500} = \frac{1}{643}$ part of a second, is the time of completing the penetration of 14 inches deep.

31. Upon the whole then it appears, that our rule will give, without sensible error, the true velocity with which the ball strikes the pendulum. But this is not, however, the same velocity with which the ball issues from the mouth of the gun, which will be indeed something greater than the former, on account of the resistance of the air which the ball passes through in its way from the gun to the pendulum. And though this space of air be but small, and though the elastic fluid of the powder pursue and urge the ball for some distance without the mouth of the piece, and so in some degree counteracts the resistance of the air, yet it will be proper to enquire into the effect of this resistance, as it will probably cause a difference between the velocity of the ball, as computed from the vibration of the pendulum and the vibration of the gun; which difference will, by-the-by, be no bad way of measuring the resistance of the air, especially if the gun be placed at a good distance from the pendulum; for the vibration of the gun will measure the velocity with which the ball issues from the mouth of it; and the vibration of the pendulum, the velocity with which it is struck by the ball.

32. To find therefore the resistance of the air against the ball in any case: it is first to be considered, that the resistance to a plane moving perpendicularly through a fluid at rest, is nearly equal to the weight or pressure of a column of the fluid, whose altitude is the height through which the body must fall, by the force of gravity, to acquire the velocity with which it moves through the fluid, the base of the column being equal to the plane. So that, if a denote the area of the plane, v the velocity, n the specific gravity of the fluid, and $h = 16{\cdot}09$ feet; the altitude due to the velocity v being $\frac{vv}{4h}$, the whole resistance, or motive force m, will be $a \times n \times \frac{vv}{4h} = \frac{anvv}{4h}$.

Now, if d denote the diameter of the ball, and $k = {\cdot}7854$, then shall $a = kd^2$ be a great circle of the ball; and consequently $m = \frac{knd^2v^2}{4h}$ = the motive force on the surface of a circle equal to a great circle of the ball.

But the resistance on the hemispherical surface of the ball is only one half nearly of that on the flat circle of the same diameter; therefore $m = \frac{knd^2v^2}{8h}$ is the motive force on the ball; and if w denote its weight, $\frac{knd^2v^2}{8hw}$ will be equal to f the retarding force.

Since $\frac{2}{3}kd^3$ is the magnitude of the sphere, if N denote its density or specific gravity, its weight w will be $= \frac{2}{3}kd^3\text{N}$; consequently the retarding force f or $\frac{m}{w}$

$$\text{will be} = \frac{knd^2v^2}{8h} \times \frac{3}{2kd^3\text{N}} = \frac{3nvv}{16dh\text{N}}.$$

But by the laws of forces $v\dot{v} = 2hf\dot{x} = \frac{-3nvv}{8d\text{N}}\dot{x}$, and $\frac{\dot{v}}{v} = \frac{-3n}{8d\text{N}}\dot{x} = -e\dot{x}$, where x is the space passed over, putting $e = \frac{3n}{8d\text{N}}$, and making the value negative, because the velocity v is decreasing. And the correct fluent of this is log. V − log. v or log. $\frac{\text{V}}{v} = ex$, where V is the first or greatest velocity of projection. Or if A be $= 2{\cdot}718281828$ &c, the number whose hyperbolic logarithm is 1, then is $\frac{\text{V}}{v} = \text{A}^{ex}$, and hence the velocity $v = \frac{\text{V}}{\text{A}^{ex}} = \text{V}\text{A}^{-ex}$. So that

the first velocity is to the last velocity, as A^{ex} to 1. And the velocity lost by the resistance of the medium is nearly $(A^{ex} - 1)v$ or $\frac{A^{ex}-1}{A^{ex}}v$.

33. Now to adapt this to the case of our balls, which weighed on a medium $16\frac{3}{4}$ ounces, when the diameter was 1·96 inches; we shall have $1{\cdot}96^3 \times {\cdot}5236 =$ the magnitude of the ball; and as 1 cubic foot, or 1728 cubic inches, of water, weighs 1000 ounces, therefore $\frac{1728 \times 16\frac{3}{4}}{1000 \times 1{\cdot}96^3 \times {\cdot}5236} = 7{\cdot}344 = N$ is the specific gravity of the iron ball; which is very justly something less than the usual specific gravity of solid cast iron, on account of the small air bubble which is within side of all cast metal balls. Also the mean specific gravity of air is ·0012, which is the value of n. Hence $e = \frac{3n}{8dN} = \frac{3 \times {\cdot}0012 \times 12}{8 \times 1{\cdot}96 \times 7{\cdot}344} = \frac{1}{2666}$.

Now the common distance of the face of the pendulum from the trunnions of all the guns, was $35\frac{1}{2}$ feet; and the distance of the muzzles of the four guns, was nearly $34\frac{1}{4}$ for the 1st or shortest gun, 34 for the 2d, 33 for the 3d, and $31\frac{1}{2}$ for the 4th. But as the elastic fluid pursues and urges the ball for a few feet after it is out of the gun, it may be supposed to counter-balance the resistance of the air for a few feet, the number of which cannot be certainly known, and therefore we shall suppose 32 feet to be the common distance, for each of the guns, which the ball passes through, before it reach the pendulum. Hence then the distance $x = 32$; and consequently $ex = \frac{32}{2666} = \frac{16}{1333}$.

Then $A^{ex} - 1 = {\cdot}01207 = \frac{1}{83}$ nearly. That is, the ball loses nearly the 83d part of its last velocity, or the 84th part of its first velocity, in passing from the gun to the pendulum, by the resistance of the air. Or the velocity at the mouth of the gun, is to the velocity at the pendulum, as 84 to 83; so that the greater diminished by its 84th part gives the less, and the less increased by its 83d part gives the greater. But if the resistance to such swift velocities as ours

be about three times as great as that above, computed from the nature of perfect and infinitely compressed fluids, as Mr. Robins thinks he has found it to be, then shall the velocity at the gun lose its 28th part, and the greater velocity will be to the less, as 28 to 27. This however is a circumstance to be discovered from our experiments, the result from which we shall hereafter have an opportunity of comparing with the conclusion in this article, when it may either be confirmed or corrected.

On the Velocity of the Ball, as found from the Recoil of the Gun.

34. It has been said, by more than one writer on this subject, that the effect of the inflamed power on the recoil of the gun, is the same, whether it be charged with a ball, or fired by itself alone; that is, that the excess of the recoil when charged with a ball, over the recoil when fired without a ball, is exactly that which is due to the motion and resistance of the ball. And this it is said they have found from repeated experiments. Now supposing those experiments to be accurate, and the deductions from them justly drawn; yet as they have been made only with small balls, and small charges of powder, it may still be doubted whether the same law will hold good when applied to such cannon balls, and large charges of powder, as those used in our present experiments. Which is a circumstance that remains to be determined from the results of them. And this determination will be easily made, by comparing the velocity of the ball as computed from this law, with that which is computed from the vibration of the ballistic pendulum. For if the law hold good in such cases as these, then the velocity of the ball, as deduced from the vibration of the gun, will exceed that which is deduced from the vibration of the pendulum, by as much as the velocity is diminished by the resistance of the air between the gun and the pendulum.

35. Taking this for granted then in the mean time, namely, that the effect of the charge of powder on the recoil of the gun, is the same either with or without a ball, it will be proper here to investigate a formula for computing the velocity of the ball from that recoil. Now, on the foregoing principle, if the chord of vibration be found for any charge without a ball, and then for the same charge with a ball, the difference of those chords will be equal to the chord which is due to the motion of the ball. This follows from the property of a circle and a body descending along it, namely, that the velocity is always as the chord of the arc described in a semivibration.

Let then c denote this difference of the two chords, that is

c = the chord of arc due to the ball's velocity,
G = weight of the gun and iron stem, &c.
b = weight of the ball,
g = distance of the centre of gravity of G,
o = distance of its centre of oscillation,
n = its n°. of oscillations per minute,
i = distance of the gun's axis, or point of impact,
r = radius of arc or chord c,
v = velocity of the ball,
V = velocity of the gun, or of the axis of its bore:

Then, because $biiv$ is the sum of the momenta of the ball, and $GgoV$ the sum of the momenta of the gun; and because action and re-action are equal, these two must be equal to each other, that is $biiv = GgoV$. But, because V is the velocity of the distance i, therefore by similar figures $i : o :: V : \frac{oV}{i}$ the velocity of the centre of oscillation. And because the velocity of this centre, is equal to the velocity generated by gravity, in descending perpendicularly through the height or versed sine of the arc described by it,

and because $2r : c :: c : \frac{cc}{2r}$ = versed sine to radius r,

and $r : o :: \frac{cc}{2r} : \frac{cco}{2rr}$ = vers. sine to radius o,

therefore $\sqrt{h} : \sqrt{\frac{cco}{2rr}} :: 2h : \frac{c}{r}\sqrt{2ho}$, the velocity

Z2

of the centre of oscillation as deduced from the chord c of the arc described, where $h = 16{\cdot}09$ feet; which velocity was before found $= \frac{o\text{v}}{i}$.

Therefore $\frac{o\text{v}}{i} = \frac{c}{r}\sqrt{2ho}$, or $o\text{v} = \frac{ci}{r}\sqrt{2ho}$.

Then this value of $o\text{v}$ being substituted in the first equation $biiv = \text{G}go\text{v}$, we have $biiv = \frac{\text{G}gci}{r}\sqrt{2ho}$, and hence the velocity $v = \frac{\text{G}gc}{bir}\sqrt{2ho} = \frac{5{\cdot}6727\,\text{G}gc}{bir}\sqrt{o}$, being the formula by which the velocity of the ball will be found in terms of the distance of the centre of oscillation and other quantities. Which is exactly similar to the formula for the same velocity, by means of the pendulum in Art. 22, using only G, or the weight of the gun, for $p + b$ or the sum of the weights of the ball and pendulum.

And if, instead of $\sqrt{o}$ be substituted its value $\sqrt{\frac{11737{\cdot}5}{nn}}$ or $\frac{108{\cdot}3398}{n}$, from Art. 20, it becomes $v = 614{\cdot}58 \times \frac{\text{G}gc}{birn}$, or $= \frac{59000}{96} \times \frac{\text{G}gc}{birn}$, the formula for the velocity of the ball, in terms of the number of vibrations which the gun will make in one minute, and the other quantities.

36. Further, as the quantities G, g, b, i, r, n commonly remain the same, the velocity will be directly as the chord c. So that, if we assume a case in which the chord shall be 1, and call its corresponding velocity u; then shall $v = cu$; or the velocity corresponding to any other chord c, will be found by multiplying that chord c by the first velocity u answering to the chord 1.

Now, by the following experiments, the usual values of those literal quantities were as follows:

viz, $\text{G} = .917$
$g = 80{\cdot}47$
$b = 1{\cdot}047$, sometimes a little more or less.
$i = 89{\cdot}15$
$r = 1000$
$n = 40{\cdot}0$, for the gun n° 2, (but the 400th part more for n° 1, and the 400th part less for n° 3, and the 200th part less for n° 4.)

Then, writing these values in the theorem, instead of the letters, it becomes $v = 12{\cdot}15c$. So that the number 12·15 multiplied by the difference between two chords described with any charge, the one with and the other without a ball, will give the velocity of the ball when the dimensions are as stated above. And when the values of any of the letters vary from these, it is but increasing or diminishing that product in the same proportion, according as the letter belongs to the numerator or denominator in the general formula $\frac{59000}{96} \times \frac{Ggc}{birn}$. When such variations happen, they will be mentioned in each day's experiments. And further, when only the values of G, g, i, n are as before specified, the same formula will become $12718 \times \frac{c}{br}$.

But note that these rules are adapted to the gun n° 2 only; therefore for n° 1 we must subtract the 400th part, and add the 400th part for n° 3, and add the 200th part for n° 4.

OF THE EXPERIMENTS.

37. We shall now proceed to state the circumstances of the experiments, for each day separately, as they happened; by this means showing all the processes for each set of experiments, with the failure or success of every trial and mode of operation; and from which also any person may recompute all the results, and otherwise combine and draw conclusions from them as occasion may require. Making but a very few cursory remarks on each day's experiments, to explain them when necessary; and reserving the chief philosophical deductions, to be drawn and stated together, after the close of the experiments, in a more connected and methodical way.

The machinery having been made as perfect as the circumstances would permit, 20 barrels of government powder were procured, all by the best maker, and numbered from 1 to 20. A great number of iron balls were also cast on purpose, very round, and their accidental asperities ground off: they were a little varied in their size and weight, but

most of them almost equal to the diameter of the bore, so as to have very little windage. The powder was uniformly mixed, and every day exactly weighed off by the same careful man, and put up in very thin flannel bags, of a size just to fit the bore of the gun; a thread was tied round close by the powder, after being shaken down, and the flannel cut off close by the thread, so as to leave as short a neck as possible to the bag. The charge of powder was pushed gently down to the lower or breech end of the bore, and the same quantity of powder always made to occupy nearly the same extent, by means of the divisions of inches and tenths marked on the ramrod. The ball was then put in, without using any wads, and set close to the charge of powder, and kept in its place by a fine thread crossed two or three times about it, which by its friction gave it a hold of the sides of the bore, as the windage was very small. The gun was directed point blank, or horizontal, and perpendicular to the face of the pendulum block, $35\frac{1}{2}$ feet distant from the trunnions, and was well wiped and cleaned out after each discharge, which was made by piercing the bottom of the charge through the vent, and firing it by means of a small tube. An account was kept of the barometer and thermometer, placed within a house adjoining, and shaded from the sun.

The machinery having been all prepared and set up in a convenient place in Woolwich Warren, the experimenters went out on the 6th of June 1783, to try the effects of them for the first time, which were as follow.

38. *Friday, June* 6, 1783; *from* 10 *till* 12 *A. M.*

The weather was warm, dry, and clear.

The barometer at 30·17, and thermometer at 60°.

The intention of this day's experiments, was to try and adjust the apparatus; to ascertain the proper distance of the pendulum; as also the comparative strength of the different barrels of powder, by firing several charges of it, without

balls or wads. Out of the 20 barrels of powder, were selected the 6 which had been found to be most uniform, and nearest alike, by the different eprouvettes at Purfleet, which were n^{os} 2, 5, 13, 15, 18, 19; of which the first two only were tried this day, as below. The gun was the short one, n° 1, and weighed this day, with the leaden weights and iron stem, 906lb: the distance of the tape, by which the chord of its recoil was measured, was not taken, and it was probably a little more than the usual length, 110 inches, employed in most of the experiments of this year.

Here it appears that the quantity of recoil increased in a higher ratio than the quantity of powder.

The pendulum was not moved by the blast of the powder in these experiments.

No. of Experim.	Powder sort	Powder weight	Chord of recoil	Medium of recoil
		oz	inches	
1	n° 2	2	2·25	2·30
2		2	2·35	
3		2	2·30	
4	n° 5	2	2·55	2·50
5		2	2·40	
6		2	2·55	
7	n° 5	8	13·00	12·88
8		8	12·75	
9	n° 2	8	12·50	12·50
10		8	12·50	

39. *Saturday, June* 7, 1783; *from* 9½ *A. M. till* 12.

The weather cloudy or hazy, but it did not rain.

Barometer 30·25, Thermometer 60°.

To try all the 6 sorts of powder, and the effect of the blast on the pendulum, when high charges are used.

The first 14 rounds were with the same apparatus and gun n° 1, as the former day. The other four rounds were with the gun n° 4, but without the leaden weights; it weighed with the iron 561lb.

These recoils are very uniform, and there appears to be but little difference in the quality of the powder among the several sorts.

No.	Powder sort	Powder weight	Recoil	Mediums
		oz	inches	
1	2	8	13·35	13·38
2		8	13·40	
3	5	8	13·50	13·28
4		8	13·05	
5	13	8	13·50	13·23
6		8	12·95	
7	15	8	13·50	13·35
8		8	13·20	
9	18	8	13·25	13·38
10		8	13·50	
11	19	8	12·95	12·95
12		8	12·95	
13	13	2	2·25	
14		16	26·00	
				vibr. of pendulum
15	13	2	4·5	0
16		4	10·8	0
17		8	24·7	0·25
18		16	53·3	1·10

All the charges were in flannel bags, except n^{os} 14 and 18, of 16 oz each, for want of bags large enough provided to put it in. Each charge was rammed with two or three slight strokes. A considerable quantity of the powder of n^{o} 14 was blown out unfired; many of the grains were found on the ground, and on the top of the pendulum block, and many were found sticking in the face of it. By the force of these striking it, and by the blast of the powder, or motion of the air, the pendulum was observed visibly to vibrate a little: but the measuring tape had not been put to it. This was therefore now added, to measure the vibration by. And, to try to what degree the pendulum would be affected by the explosion of the powder, the 7 feet amusette was suspended,

and pointed opposite the centre of the pendulum for the last 4 rounds. The pendulum was accordingly observed to move a little with the 8 ounces, but more with the 16 ounces, as appears at the bottom of the last column of the table above. The pendulum being thus much affected, we were convinced of the necessity of making a paper screen to place between the gun and the pendulum; which we accordingly did, and used it in the whole course of experiments, at least in the larger charges. At the last charge, which was 16 ounces of loose powder, much fewer grains were blown out than with the like charge at n° 14 with the short gun. The recoil at n° 14 is evidently less than it ought to be; owing to the quantity of unfired powder that was blown out. It is remarkable that the recoil of the two guns, with the same charge, both for 2 ounces and 8 ounces, are nearly in the reciprocal ratio of the weights of the guns; a small excess only, over that proportion, taking place in favour of the long gun, as due to its superior length. The recoils are, all of them, visibly in a higher proportion than the charges of powder: for, in the last four experiments, the charges of 2, 4, 8, 16 ounces, are in the continued proportion of 1 to 2; while their recoils 4·5, 10·8, 24·7, 53·3, are all in a higher ratio than that of 1 to 2; for, dividing the 2d by the 1st, the 3d by the 2d, and the 4th by the 3d, the three successive quotients are 2·40, 2·29, 2·16; which are all above the double ratio, but approximating, however, towards it, as the charge is increased. And further, if we divide these quotients successively one by another, the two new ratios or quotients will be nearly equal. So that, ranging those recoils in a column under each other, and their two successive orders of ratios in the adjacent columns, we shall have in one view the law which they observe, as here below, where they always tend to equality.

4·5			
10·8	2·40		
24·7	2·29	·954	
53·3	2·16	·944	·99

Again, if we take successive differences between the same recoils, and between these differences, and then between the second differences, and so on, thus

$$\begin{array}{c|c|c|c}
4\cdot5 & & & \\
 & 6\cdot3 & & \\
10\cdot8 & & 7\cdot6 & \\
 & 13\cdot9 & & 7\cdot1 \\
24\cdot7 & & 14\cdot7 & \\
 & 28\cdot6 & & \\
53\cdot3 & & &
\end{array}$$

the columns, as well as the lines, descending obliquely from left to right, have their numbers approaching, and at length ending, in the ratio of 2 to 1, the same as the quantities of powder.

40. *Friday, June* 13, 1783 ; *from* 11 *till* 1 *o'clock*.

The air moist, with small rain at intervals.

The gun n° 2 was mounted, and loaded with all the leaden weights: it was charged with the following quantities of powder ; sometimes with a ball, and sometimes without one, as denoted by the cipher 0, in the columns of weight and diameter of ball. The radius to the tape was 110·2. As these experiments were made only to discover if the leaden weights would render the gun sufficiently heavy, that the recoil might not be too large with the high charges of powder and ball, the pendulum block was removed, to let the balls enter and lodge in the bank which was behind it.

No.	Powder		Ball's			Recoil
	sort	wt	diameter	wt		
		oz	inches	oz	dr	inches
1	19	2	0	0		2·5
2	19	2	0	0		2·5
3	19	4	0	0		5·2
4	19	8	0	0		13·5
5	19	16	0	0		28·1
6	18	16	0	0		28·0
7	19	2	1·9	15	4	8·9
8	19	4	1·9	15	4	16·15
9	19	8	1·9	15	4	26·5
10	19	16	1·9	15	4	41·75
11	18	16	1·9	15	4	34·3
12	18	16	1·9	15	4	35·15
13	19	16	1·975	16	13	36·0
14	19	16	1·965	16	13	33·5

Here again it appears that the recoils, without balls, are always in a greater ratio than the charges of powder. It also appears that the recoils, when balls are employed, are nearly in the ratio of the quantities of powder, when the charges are small; but gradually decreasing more and more below that ratio, as the charge of powder is increased. And if we subtract each recoil without a ball, from the corresponding recoil with a ball, for the same charge of powder, taking the differences as here below,

Weight of powder	oz 2	4	8	16
Recoils with a ball	8·9	16·2	26·5	34·7
Recoils without -	2·5	5·2	13·5	28·0
Differences -	6·4	11·0	13.0	6·7

it will appear that those differences increase as far as to the charge of 8 ounces, and then decrease again.

There must have been some mistake in the 10th round, as the recoil, which is 41·75 inches, is greater than can well be expected with that charge of powder. Probably the tape had entangled, and been drawn farther out in the return of the gun from the recoil.

41. *Monday, June* 23, 1783.

We went with the workmen, and took the weight and dimensions of the several parts of the machinery, both of the pendulum with its stem, and of the guns with their frame or iron stem, and the leaden weights to fit on about the trunnions.

OF THE PENDULUM.

Total weight with all the iron work about it - 559 lb,
Distance from its axis to the centre of gravity 75·2 inch.
Ditto - - - to the tape or lowest point 115·1 inch.
Ditto - - - to the top of the block 76·3 inch.
Dimensions of the wooden block - 18, 22, and 24 inch.
that is, breadth of the face 18, height of the face 24, and length from front to back 22.

THE GUN FRAME OR STEM.

Total weight of all the iron work	- -		188 lb,
Dist. from its axis to the cen. of grav. (without gun)			44·25 inch.
Ditto	- -	to the tape or lowest point	110 inch.
Ditto	- -	to centre of the trunnions	90·3 inch.
Ditto	- -	to the perpendicular arm	75·75 inch.

The following figure is a side-view of the gun-frame or stem, as it hung on its axis with the gun;

A being the point through which the axis passes,

G the point in the stem where it rests in equilibrio, showing the distance AG of the centre of gravity below the axis,

G g C perpendicular to A G,

A P a plumb-line cutting G C in g,

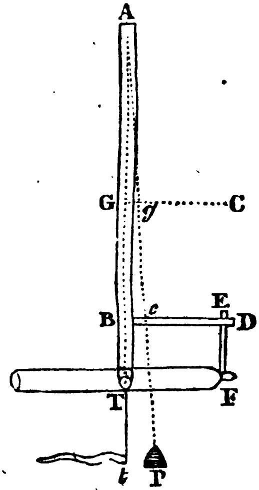
g the centre of gravity of the iron work,

B D a fixed perpendicular arm,

E F a sliding piece to support the gun,

T the centre of the trunnions,

t the place of the tape or lowest point.



And the dimensions or measures to these points are as follow:

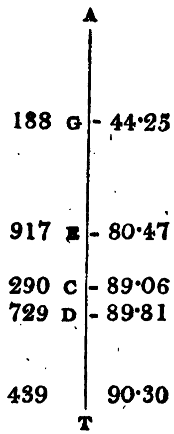
		inches.			inches.
A G	- -	44·25	A t	- -	110·0
A B	- -	75·75	B e	- -	5·6
A T	- -	90·3	G g	- -	3·3

Breadth of stem A T - - 3·5, and from the middle of this breadth the distances B e and G g are measured.

42. The following are also the measurements taken to settle the position of the compound centre of gravity of the gun, with its leaden weights and iron stem, all together.

No. of the gun	Diameter of the trunnions	Diameter of the gun at centre of trunnions	Centre of gravity or axis of gun alone: behind the muzzle	behind the centre of trunnions	above cent. of trunnions	below axis of vibration	Centre of gravity of the whole below axis
1	2·2	7·00	18·5	1·4	1·24	89·06	80·47
2	2·2	5·89	24·5	1·8	1·24	89·06	80·47
3	2·25	5·06	37·4	4·2	1·11	89·19	80·50
4	2·2	4·84	51·3	3·0	1·02	89·28	80·44

The numbers in the last column of this table, are the values of the letter g, in the formula for the velocity by means of the recoil of the gun. This letter may always be supposed to have the value 80·47 inches, as the two last numbers of the column differ from it but ·03 only, or about the 2700 part of the whole, inducing an error of only about half a foot in the velocity of the ball. The values of g, in this last column of the table, were computed in the following manner.



Suppose the centres of gravity of the stem, gun, and leads, to be all reduced to the line of the stem A T, connecting the axis and centre of the trunnions T, in which they are situated very nearly;
so that G be the centre of grav. of the stem,
C - - - - - - - of the gun,
T - - - - - - - of the leads;
also the numbers on the right-hand side of these points, namely, 44·25 and 89·06 and 90·3, are the measured distances below the axis at A; and the numbers on the left-hand side, namely, 188, 290, and 439, are the weights of the bodies belonging to those centres of gravity. Then, from the property of the centre of gravity, we shall have these operations;

439		90·30			
290		89·06			
729 :	290 : :	1·24 :	0·49	- -	TD
			90·30	- -	AT
			89·81	- -	AD
			44·25	- -	AG
729					
188			45·56	- -	GD
917 :	729 : :	45·56 :	36·22	- -	GE
			44·25	- -	AG
			80·47	- -	AE

where D is the centre of gravity of the bodies at the points C and T, or of the gun and leads; and E the centre of gravity of the two bodies placed at the points G and D, or of all the three bodies at the points G, C, T; that is, E is the compound centre of gravity for both gun, iron, and leads, in one mass. And the same operation is to be repeated for the other guns.

43. It may here be also remarked, that the mean number of vibrations per minute, for every gun, weighing in all 917lb, taken among the actual vibrations of each day, is for

n° 1	n° 2	n° 3	n° 4
40·1	40·0	39·9	39·8

which number must be used as the true value of n, in the formula for the velocity of the ball by means of the recoil of the gun. The number of the gun's vibrations was commonly tried every day, and they were found to vary but little, and among them all the numbers above-mentioned, are the arithmetical mediums.

44. Moreover, the mean numbers for the pendulum, among all the daily measurements of its weight, centre of gravity, and oscillations per minute, are thus:

weight	g	n.
660 lb	77·3	40·2

Of the great number of these measures that were taken, the variations among them were sometimes in excess

and sometimes in defect; and therefore the above numbers, which are the means among the whole, as long as the iron work remains the same, will probably be very near the truth. And by using always these, with proportional alterations in g and n, for any alteration in the weight p, the computations of the velocity of the ball will be made by a rule that is uniform, and not subject at least to accidental single errors. When the weight of the pendulum varies by the wood alone of the block, or the straps about it, the alteration is to be made at the centre of the block, which is exactly 88·3 inches below the axis; that is, in that case the value of i is 88·3 in the formula $\frac{i-g}{p+b}b$, or the correction for g; and in $\frac{bin(inn-140850)}{140850(2pg+bi)+biinn}$, the correction of n. But when the alteration of the weight p arises from the balls and plugs lodged in the same block, then the value of i in those corrections is the medium among the distances of the point struck. And when the iron work is altered, the middle of the place altered gives the value of i in the same theorems.

In these corrections too, p denotes 660, g 77·3, n 40·2, and b the difference between 660 and any other given weight of the pendulum; which value of b will be negative when this weight is below 660, otherwise positive; so that $p+b$ is always equal to this weight of pendulum.

And if these values of p, g, n be substituted for them in those corrections, they will become

$$\frac{i-77\cdot3}{660+b}b, \text{ or } \frac{i-77\cdot3}{p}b, \text{ the correction for } g, \text{ and}$$

$$\frac{40\cdot2bi(i-87\cdot1)}{8887336+bi(i+87\cdot1)} \text{ the correction for } n.$$

And further, when $i = 88\cdot3$, the same become

$$\frac{11b}{660+b} \text{ or } 11 - \frac{7260}{p} \text{ the correction for } g, \text{ and}$$

$$\frac{b}{2086+3\cdot6b} \text{ or } \cdot4545 - \frac{160}{p-80} \text{ the correction for } n,$$

as adapted to an alteration at the centre of the pendulum.

And, in that case,

$$G = 88\cdot3 - \frac{7260}{p} \text{ is the new value of } g, \text{ and}$$

$$N = 39\cdot923 + \frac{160}{p-80} \text{ is the new value of } n.$$

But these corrections will have contrary signs when b is negative, as well as the second term in each of the denominators.

45. *Monday, June* 30, 1783; *from* $9\frac{1}{2}$ *A. M. to* $2\frac{1}{2}$ *P. M.*

The air clear, dry, and hot.

Barometer 30·34, and Thermometer 74.

We began this day, for the first time, to fire with balls against the pendulum block. The powder of the six barrels before-mentioned, had been all well mixed together for the use of the experiments, that they might be as uniform as possible, in that, as well as in other respects.

The Gun was n° 1, with the leaden weights.

Its weight and the distance of its centre of gravity, were as before-mentioned; the distance of the tape it was forgotten this day to measure, but from circumstances judged to be $106\frac{1}{2}$.

Pendulum. Its weight - - - $559 = p$,
Distance to the tape $115{\cdot}1 = r$.

N°	Powder.	Ball's wt		Ball's diam	Vibration of gun	Vibration of pend	Struck below axis	Plugs	Values of p	Values of g	Values of n	Veloc. of the ball
	oz	oz	dr	inches	inches	inches	inches	in	lb	inches		feet
1	2				2							
2	2				2							
3	8				11·2							
4	16				23·4							
5	16	16	13	1·95	34·	23	87·9		559·0	75·30	40·30	1392
6	16	16	13	1·95	35·4	25	86·8		560·1	75·32	40·30	1534
7	16	16	13	1·95	34·8	23·7	88·8		561·1	75·35	40·29	1426
8	16	16	13	1·95	35·1	25	87·6	15	562·2	75·37	40·29	1530
9	16	16	13	1·95	35·7	23·7	88·2	10	564·6	75·39	40·28	1445
10	16	16	13	1·95	35·2	23·1	88·3		566·5	75·42	40·28	1412
				medium	35·0					medium		1456

The first 4 rounds were with powder only; the other 6 with balls, all of the same size and weight.

The diameter of the gun bore being	2·02,	and
the diameter of the ball - - -	1·95,	consequently
the windage was - - - - - -	0·07	
Mean length of the charge of powder	10·6	

The two oaken plugs which were driven in, to fill up the holes, after the 8th and 9th rounds, weighed about $1\frac{1}{4}$ oz. to each inch of their length. The whole weights of these plugs, and the weights of the balls lodged in the block, were continually added to the weight of the pendulum, to complete the numbers for the values of p in the 9th column; and from these numbers the correspondent values of g and n, in the next two columns, are computed by their proper corrections in Arts. 23 and 25. After which, the velocities contained in the last column are computed by the formula in Art. 24. And the medium among all these velocities, as well as that of the recoils of the gun, are placed at the bottom of their respective columns.

From the mean recoil with ball	35·0
take the recoil without a ball	23·4
there remains - - - - -	$c = 11·6$

Then, having $b = 1·051$, and $r = 106·5$, by the rule $12718 \times \frac{c}{br}$ in Art. 36, we have only 1315 feet, for the velocity of the ball as deduced from the recoil of the gun; which is 141 less than the velocity found by the vibration of the pendulum, or about $\frac{1}{10}$th of the whole velocity.

The powder blown out unfired was not much. The apparatus performed all very well, except only that the wood of the pendulum seemed not to be very sound, as it was pierced quite through by the end of this day's experiments; though the sheet lead with which the back was covered, as well as the face, just prevented the balls and pieces of the wood from falling out at the back of the pendulum.

46. *Saturday, July 5, 1783; from 9 till 2 o'clock.*

The weather clear, dry, and hot.

Barometer 30·27, and Thermometer 74.

GUN, n° 3.		PENDULUM.	
Weight - - - -	917	Weight - - - -	846
To centre of gravity	80·47	To centre of gravity	79·6
To the tape - - -	109·7	To the tape - - -	117·3

N°	Powder	Ball's		Chord of vib.		Point struck	Plugs	Values of			Veloc. of the ball
		wt	diam	gun	pend			p	g	n	
	oz	oz dr	inches	inches	inches	inches		lbs	inches		
1	2			2·3							
2	2			2·3							
3	8			13·0							
4	8			14·1							
5	8			13·6							
6	16			26·3							
7	16			28·7	0·5						
8	16			26·5	0·3						
9	16	16 13	1·95	39·0	24·2	89·0		846·0	79·3	41·48	

A large piece had been cut out of the middle part of the pendulum, from the face almost to the back, to clear away the damaged part of the wood; and the vacuity was run full of lead, from an idea that the pendulum would not so soon be spoiled, and consequently that it would need less repairs. But this did not succeed at all; for the only shot we discharged, namely, n° 9, would not lodge in the lead, but broke into a thousand small pieces, many of which stuck in the lead, and formed a curious appearance; but the greater number rebounded back again, to the great danger of the bystanders. The ball made a large round excavation in the face of the lead, of 5 inches diameter in the front, and 3¼ inches deep in the centre of the hole.

Length of the charge of 16oz was 11 inches.

47. *Friday, July* 11, 1783; *from* 9 *A. M. till* - - -

Fine, clear, hot weather.

GUN, n° 3.		PENDULUM.	
Weight - - - -	917	Weight - - - -	610
To centre of gravity	80·47	To centre of gravity	76·4
To the tape - - - -	110	To the tape - - -	118

N°	Powder	Ball's wt	Ball's diam	Chord of vib. gun	Chord of vib. pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. of the ball
	oz	oz dr	inches	inches	inches	inches					
1	2			2·5							
2	2			2·5							
3	16			28·4							
4	16			25·7							
5	16			28·3							
6	16	16 13	1·95	44·6	34·0	89·1					

Length of the charge of 16 oz was 11·2 inches.

The pendulum had been altered since the former day. The core of lead being taken out, some layers of rope were laid at the bottom of the hole, then the remainder up to the front filled with a piece of sound elm, and the face covered with sheet lead.

At the last round, or that with ball, the iron tongue which held the tape of the pendulum, having slipped down by the loosening of a screw, was strained and bent. Which stopped the experiments till it could be repaired.

48. *Saturday, July 12, 1783; from 9 A. M. till* - - -

Fine, clear, hot weather.

The pendulum, gun n° 3, and apparatus, were in every respect the same as in the last day's experiments, excepting that the radius of the tape, in the gun, was 110·2 inches instead of 110.

N°	Powder.	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches		lb	inches		feet
1	2			2·5							
2	2			2·5							
3	16			28·0							
4	16			29·0							
5	16			28·0							
6	16	16 13	1·96	44·1	33·7	89·6		607·0	76·34	40·25	2151
7	16	16 13	1·96	42·6	30·9	90·3		608·1	76·36	40·25	1960
8	16	16 13	1·96	46·8	32·3	89·3		609·1	76·38	40·24	2076
9	16	16 13	1·96	44·4	30·5	89·6		610·2	76·39	40·24	1958
10	16	16 13	1·96	43·9	31·4	89·2		611·2	76·41	40·24	2028
11	16	16 13	1·96	42·3	31·5	90·7		612·3	76·43	40·23	2005
			medium	44·0						medium	2030

The mean length of the charge of 16 oz was 11·7 inches. But this height was always taken when the cartridge was uncompressed: so that the powder lay looser than in former experiments. By a small pressure it occupied about $\frac{1}{4}$ of an inch less space.

The value of p at beginning this day is made a little less than the pendulum weighed at first, for reasons to be mentioned hereafter.

The mean recoil with a ball is 44·0, and without a ball 28·5, the difference of which is $15·5 = c$. Also, in the formula for the velocity by means of the gun, we have $b = 1·051$, and $r = 110·2$. Consequently $v = \frac{401}{400} \times 12718 \times \frac{c}{br}$.

= 1706 for the velocity by that method. But the mean velocity by the pendulum is 2030, which exceeds the former by 324, or almost $\frac{1}{6}$ of the whole velocity.

49. *Thursday, July* 17, 1783; *from* 12 *till* 3 *P. M.*

Fine, clear, hot weather.

Barometer 30·23, Thermometer 72° at 9 o'clock.

GUN n° 1.

Weight - - - -	917
To centre of gravity	80·47
To the tape - - -	110·2

It swung very freely, and would have continued its vibrations a long time; owing to the ends of the axis being made to turn or roll upon a convex iron support, and kept from going backward and forward, with the vibrations, by two upright iron pins, placed so as not quite to touch the axis, but at a very small or hair-breadth distance from it.

PENDULUM.

Weight - - - -	657
To centre of gravity	77·26
To the tape - - -	118

The pendulum would not vibrate longer than 1 minute before the arcs became imperceptible, owing to the friction of the upright pins, which touched and bore hard against the sides of the axis, unlike those of the gun, though they had the same kind of round support to roll on. The pendulum had been well repaired, and strengthened with iron bars, and straps going round it in several places, except over the face. Also thick iron plates were let into, and across it, near the back part, then over them was laid a firm covering of rope, after which the rest of the hole was filled up with a block of elm, and finally the face covered over with sheet lead.

No	Powder	Ball's wt	Ball's diam	Vibration of un	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	in	lb	inches		feet
1	2			2·3							
2	2			2·4							
3	8			12·3							
4	8			13·8							
5	8			11·9							
6	8			13·0							
7	8			13·2							
8	8	16 13	1·96	26·3	20·5	88·6	10	657·0	77·26	40·20	1450
9	8	16 13	1·96	27·2	21·9	89·7	9	658·5	77·28	40·20	1534
10	8	16 13	1·96	25·9	20·3	89·9	9	660·1	77·30	40·20	1423
11	8	16 13	1·96	26·8	20·5	88·6	8	661·6	77·33	40·20	1462
12	8	16 13	1·96	26·3	20·4	88·5	8	663·2	77·35	40·20	1460
13	8	16 13	1·96	26·8	20·9	88·6	8	664·7	77·37	40·20	1497
										mean	1471

The mean length of the charge of 8 oz was 5·9.

The pendulum, having been so well secured, suffered but little by this day's firing, only bulging or swelling out a little at the back part. All the balls were left in it, and all the holes were successively plugged up with oaken pins of near 2 inches diameter, which weighed 11 oz to every 10 inches in length.—The arcs described, both by the gun and pendulum, are pretty regular. And the whole forms a good set of experiments.

The mean recoil of the gun with ball 26·55
without ball 12·84
difference $c =$ 13·71

Then $v = \frac{199}{200} \times 12718 \times \frac{c}{br} = \frac{399}{400} \times 12718 \times \frac{13{\cdot}71}{1{\cdot}051 \times 110{\cdot}2} =$ 1501, the velocity of the ball as deduced from the recoil of the gun; which exceeds that deduced from the pendulum by 30, or nearly $\frac{1}{49}$th part of this latter.

50. *Friday, July* 18, 1783; *from* 9 *A. M. till* 12.

Fine and warm weather.

Barometer 30·28, and Thermometer 68° at 9 A. M.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	in	lb	inches		feet
1	2			2·5							
2	2			2·55							
3	4			6·45							
4	4			6·05							
5	8			13·8							
6	8			13·9							
7	8			13·55							
8	8	16 13	1·96	28·35	24·35	87·8	9	664·7	77·37	40·19	1764
9	8	16 13	1·96	28·7	24·35	88·0	8	666·3	77·40	40·19	1765
10	8	16 13	1·96	28·3	24·3	87·9	7	667·8	77·42	40·19	1768
11	4	16 13	1·96	18·3	18·9	87·8	6	669·4	77·44	40·19	1380
12	4	16 13	1·96	18·0	18·4	87·3	6	670·9	77·47	40·19	1352
13	4	16 13	1·96	16·7	16·8	87·8	5	672·5	77·49	40·19	
14	4	16 13	1·96	18·4	18·0	87·7	4	674·0	77·51	40·19	1327

A fresh barrel of the mixed powder was opened for use this morning; and in the first 7 rounds, which were with powder only, that of the old and new barrel were used alternately, but no difference was observed.—The length of the charge of 4 oz was 3·2, and that of 8 oz was 5·9 inches.

The Gun was n° 3.——Its weight 917 -
To centre of gravity 80·47
To the tape - - - 110

It swung so freely, that after many hundred vibrations the arcs were scarce sensibly diminished. This gun heated more at the muzzle than n° 1 did, being much thinner in metal there: but it was never very hot to the hand in that part, and very little indeed about the place of the charge; for the heat was gradually less and less all the way from the muzzle to the breech, where it was not sensible to the hand.

The PENDULUM. Its weight at first round 664·7
To the tape - - - - 117·8

It had remained hanging since the last day's experiments, with all the balls and plugs in it, which increased its weight by 10 lb, except an allowance for evaporation, and increased the distance of the centre of gravity by little more than $\frac{1}{10}$th of an inch. It vibrated with great freedom ; for it had this day been made to turn very freely on its axis, by placing the upright pins, which confine it sideways, so as not quite to touch the axis, like those of the gun yesterday ; and the effect was very great indeed, for it appeared as if it would have vibrated for a great length of time ; whereas on the former days it stopped motion in about 1 minute, or at least after that the arcs soon became too small to be counted.—By this day's firing the pendulum seemed not to be much injured, the back part not appearing to be altered, and the fore part only a little swelled out, the piece of wood which had been fitted in there, starting a little forward, and bulging out the facing of lead.

Of the plugs every 10 inches in length weighed 11 oz.

	4 oz	8 oz
The mean recoil of gun with ball	18·23 -	- 28·45
without	6·25 -	- 13·75
the difference or c =	11·98 -	- 14·70
Hence the velocity by the recoil is	1321 -	- 1620
Mean ditto by the pendulum -	- 1353 -	- 1766
Which exceeds that by recoil by	32 -	- 146
Or the	42d -	- 12th part.

This appears to be a good set, being very uniform, except the 13th round, which has been omitted, as evidently defective in the arc described both by the gun and pendulum, from some undiscovered and unaccountable cause.

51. *Saturday, July* 19, 1783 ; *from* 9 *till* 3.

A fine and warm day.

Barometer 30·12, Thermometer 70° at 9 o'clock.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	in.	lb	inches		feet
1	2			2·3							
2	2			2·4							
3	4			5·8							
4	4			5·8							
5	4			5·8							
6	4	16 13	1·96	15·9	14·8	89·8	5	674·0	77·51	40·19	1065
7	4	16 13	1·96	16·3	15·4	89·8	5	675·3	77·54	40·19	1111
8	4	16 13	1·96	16·4	15·8	90·2	5	676·6	77·56	40·19	1137
9	4	19 13	1·96	16·3	15·4	89·3	5	677·9	77·58	40·19	1122
10	2	16 13	1·96	9·8	10·7	89·1	3	679·2	77·61	40·19	783
11	2	16 13	1·96	9·9	11·0	89·5	2	680·5	77·63	40·18	803
12	2	16 13	1·96	10·1	11·1	90·3	3	681·8	77·66	40·18	805
13	2			2·55							
14	2			2·55							
15	2	16 13½	1·96	11·1	12·6	88·8	3	683·1	77·68	40·18	930
16	2	16 13½	1·96	10·7	12·0	89·4	3	684·4	77·71	40·18	881
17	2	16 12½	1·96	11·0	12·3	89·8	3	685·7	77·73	40·18	901
18	2	16 12½	1·96	10·8	11·9	89·0		687·0	77·76	40·18	881

Of the plugs every 10 inches weighed 11 ounces.

Length of the charge of 2 oz was 1·7 ; and that of 4 oz was 3·2.

The Gun was n° 1 for the first 12 n°s, and n° 3 for the rest; in order to complete the comparison between these two guns with 2, 4, 8, and 16 oz of powder. The radius

to the tape 110 inches, and the other circumstances as before.

The PENDULUM had been left hanging since yesterday, and the radius to the tape was 117·8 as before. It became however so full of balls and plugs to-day, that no more plugs could be driven in, all the iron straps being bent and forced out to their utmost stretch. It was therefore ordered to be gutted and repaired.

This is a good set of experiments; all the apparatus having performed well; and the arcs described, both by the gun and pendulum, being very uniform.

	Gun 1		Gun 3
	2 oz	4 oz	2 oz
Mean recoil with ball -	9·93	16·23	10·90
Ditto without - - -	2·35	5·80	2·55
The difference or $c =$	7·58	10·43	8·35
Hence velocity by recoil	832	1145	921
Mean ditto by pendulum	797	1109	898
Which are below recoil	35	36	23
Or nearly the part - -	$\frac{1}{23}$	$\frac{1}{31}$	$\frac{1}{39}$

52. *Wednesday, July* 23, 1783; *from* 10 *till* 3.

Fine weather.

Barometer 29·85, Thermometer 70° at 4 P. M.

N°	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches	inches	in.	lb	inches		feet
1	2				2·5							
2	2				2·5							
3	4				5·7							
4	4				5·7							
5	8				13·0							
6	8				12·9							
7	16				25·9							
8	16				26·6							
9	16				24·9							
10	16				26·5							
11	16	16	13½	1.96	39·9	22·4	87·7	12	690·0	77·78	40·18	1693
12	16	16	12½	1·96	39·0	21·3	88·3	12	691·6	77·80	40·18	1610
13	16	16	13½	1·96	41·2	23·1	88·7	9	693·2	77 82	40·18	1736
14	16	16	12½	1·96	38·3	21·2	88·7	9	694·9	77·85	40·18	1603
15	8	16	13½	1·96	27·9	21·4	89·2	8	696·5	77·88	40·18	1607
16	8	16	12½	1·96	27·1	20·2	88·6	7	698·1	77·90	40·17	1538
17	8	16	13½	1·96	27·3	20·5	88·9	8	699·8	77·92	40·17	1553
18	4	16	13½	1·96	16·8	15·1	88·0	7	701·4	77·94	40·17	1159
19	4	16	13½	1·96	16·7	14·7	88·3	6	703·0	77·96	40·17	1127
20	4	16	13½	1·96	16·6	14·6	88·5	5	704·7	77·99	40·17	1120
21	2	16	13½	1·96	9·5	9·4	88·6	4	706·3	78·01	40·16	722
22	2	16	13½	1·96	10·7	10·9	87·8	6	707·9	78·03	40·16	847
23	2	16	13½	1·96	9·6	9·4	88·5	5	709·6	78·05	40·16	727
24	2	16	13½	1·96	10·8	11·2	87·7		711·2	78·08	40·16	878

Length of charge of 2 oz was 2·1 inches
4 - - 3·3
8 - - 6·1
16 - - 10·9

Of the plugs every 10 inches weighed 12 ounces.

The Gun was nº 2.——Its weight - - - 917
To centre of gravity 80·47
To the tape - - 110
Oscillation per min. 40·6, as before
It heated very little by firing.

The Pendulum.——Its weight - 690
To the tape 117·8

It had been gutted, and repaired, by placing a stratum of lead, of two inches thick, before the iron plate, then the lead was covered with a block of wood, and the whole faced with sheet lead.

	2 oz	4 oz	8 oz	16 oz
Mean recoil with ball -	10·15	16·7	27·43	39·60
Ditto without - - -	2·50	5·7	12·95	25·98
The difference or $c =$	7·65	11·0	14·48	13·62
Hence velocity by recoil	840	1207	1592	1499
Mean ditto by pendulum	793	1135	1566	1660
Difference - - - -	+47	+72	+26	−161
Or the part - - -	$\frac{1}{17}$	$\frac{1}{16}$	$\frac{1}{60}$	$\frac{1}{10}$

So that the recoil gives the velocity with 2, 4, and 8 ounces of powder greater, but with 16 ounces much less, than the velocity shown by the pendulum.

53. *Monday, July* 28, 1783; *from* 10 *till* 2.

A very hot day.

Barometer 29·74; Thermometer 77° at 10 A. M.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz			inches							
1	2			2·6							
2	2			2·45							
3	2			2·4							
4	2			2·45							
5	2			2·7							
6	2			2·65							
7	2			2·6							
8	4			6·3							
9	4			6·35							
10	4			6·35							
11	8			13·8							
12	8			14·0							
13	8			14·1							
14	16			28·1							
15	16			27·9							

	2 oz	4 oz	8 oz	16 oz
Mean length of charge	1·9	3·3	6·2	11·0
Mean recoil of gun -	2·65	6·33	13·97	28·0
Ditto with greater wt	2·48			

The GUN n° 4.——Its weight in first 4 rounds 1003
Ditto in all the rest - - 917
Other circumstances as before.

The gun was very hot before firing, with the heat of the sun. But heated little more with firing. It was hottest at the muzzle, where the hand could not long bear the heat of it.

The PENDULUM had been gutted and repaired since the last day.

It weighed - - 702
To the tape - - 117·8
No balls were fired this day.

54. *Tuesday, July* 29, 1783; *from* 12 *till* 3.

A fine and warm day.

Barometer 29·90; Thermometer 72° at 10 A. M.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	inc	lb	inches		feet
1	2			3·0							
2	2			2·7							
3	2			2·8							
4	2			2·75							
5	4			6·45							
6	4			6·25							
7	4			6·35							
8	8			14·4							
9	8			14·3							
10	8			14·5							
11	16			29·15							
12	16			28·25							
13	16			29·2							
14	16			28·3							
15	8	16 13	1·96	29·6	25·8	89·5	12	700·0	77·92	40·17	1946
16	8	16 13	1·96	29·1	25·0	89·0	11	702·0	77·95	40·17	1902
17	8	16 13	1·96	29·1	25·8	89·5	10	704·0	77·98	40·17	1959
18	16	16 13	1·96	44·5	29·1	89·4	13	706·0	78·01	40·16	2219
19	16	16 13	1·96	43·0	27·0	89·7	9	708·0	78·03	40·16	2058
20	16	16 12½	1·96	43·6	28·8	89·7	9	710·0	78·05	40·16	2207

Of the plugs every 10 inches weighed 13¼ ounces.

The GUN n° 4.—Its weight and other circumstances as usual. It did not become near so hot as yesterday.

The PENDULUM was as weighed and measured yesterday, having hung unused.

The tape drawn out in the last three rounds, both of the gun and pendulum, was rather doubtful, owing to the wind blowing and entangling it.

	2 oz	4 oz	8 oz	16 oz
Mean length of charge -	1·8	3·4	5·6	10·8
Mean recoil with ball -	.	.	29·27	43·70
Ditto without - - - -	2·81	6·35	14·40	28·72
Difference or $c =$ - - -	.	.	14·87	14·98
Hence velocity by recoil	.	.	1643	1656
Mean ditto by pendulum	.	.	1936	2161
Difference, very great -	.	.	293	505
Or the part - - - -	.	.	$\frac{1}{7}$	$\frac{1}{4}$

55. *Wednesday, July* 30, 1783; *from* 10 *till* 12.

A fine day, moderately warm.

Barometer 30·06; Thermometer 69° at 12 o'clock.

N°	Powder	Ball's wt		Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz	dr	inches	inches	inches	inches	inc	lb	inches		feet
1	2				2·7							
2	2				2·6							
3	4				6·2							
4	4				6·0							
5	4	16	13	1·96	17·8	17·8	88·7	8	709·8	78·04	40·16	1376
6	4	16	13	1·96	17·8	17·4	86·7	9	711·3	78·07	40·16	1380
7	4	16	13	1·96	17·75	17·3	87·2	10	712·9	78·10	40·16	1368
8	2	16	13	1·96	11·0	12·2	87·8	7	714·4	78·12	40·15	960
9	2	16	12	1·96	11·25	12·4	86·8	7	715·9	78·15	40·15	993
10	2	16	12	1·96	10·9	11·9	87·3	5	717·5	78·18	40·15	951

The GUN was again n° 4, and every circumstance about it as before.

The PENDULUM the same as left hanging since yesterday, with the addition of the balls and plugs in it.

This day's experiments a good set.

	2 oz	4 oz
Mean length of charge - -	1·7	3·24
Mean recoil with ball - -	11·05	17·78
Ditto without - - - - -	2·65	6·10
Difference or c - - - - -	8·40	11·68
Hence velocity by the recoil	929	1295
Mean ditto by the pendulum	968	1375
Difference, gun less - -	39	80
Or the part - - - - - -	$\frac{1}{24}$	$\frac{1}{17}$

56. *Thursday, July* 31, 1783; *from* 10 *till* 12.

Fine warm weather.

Barometer 30·3; Thermometer 69° at 10 A. M.

No	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	in	lb	inches		feet
1	2			2·5							
2	16			23·8							
3	16			25·9							
4	16			23·8							
5	16			23·5							
6	16	16 13	1·96	37·3	17·8	89·6	6	717·2	78·18	40·15	1379
7	16	16 13	1·96	37·3	18·9	90·5	6	718·6	78·20	40·15	1453
8	16	16 13	1·96	34·5	16·4	90·2	6	720·1	78·22	40·15	1268
9	12	16 13	1·96	31·7	17·7	89·2	5	721·6	78·24	40·15	1387
10	12	16 12½	1·96	33·2	18·9	89·8	8	723·0	78·26	40·14	1477
11	12	16 12½	1·96	30·8	17·5	89·8	8	724·5	78·28	40·14	1371
12	12			21·0							
13	12			18·3							
14	12			18·8							

The GUN n° 1.—Weight and every thing else as usual.—The annular leaden weights, which fit on about the trunnions, have gradually been knocked much out of form by the shocks of the sudden recoils; so that, not fitting closely, they are subject to shake, a circumstance which probably has occasioned the irregularities in the recoils of this day.

The PENDULUM continued hanging still. It is suspected that its vibrations are not to be strictly depended on with the high charges of powder; owing to the striking of the balls against the iron plate within the block, and so perhaps causing them to rebound within it, and disturb the vibrations, which are not regular this day. After it was taken down, the pendulum was found to weigh 726 lb. But, from the weight of the balls and plugs lodged in it, it ought to have weighed 732 lb. It is therefore likely that the 6 lb had been lost, by evaporation of the moisture, in the 4 days, which is $1\frac{1}{2}$ lb per day. At the beginning of each day's experiments therefore $1\frac{1}{2}$ lb is deducted from the weight of the pendulum, or 2 lb before each of the last three days. And the like was done on some former days, for the same reason, when it appeared necessary.

Of the plugs, 10 inches weighed 10 ounces.

	12 oz	16 oz
Mean length of the charge -	8·4	11·1
Mean recoil with ball - - -	31·9	36·4
Ditto without - - - - - -	19·4	24·25
Difference, or c = - - - -	12·5	12·15
Hence velocity by the recoil	1374	1334
Mean ditto by the pendulum	1412	1367
Difference, the gun less - -	38	33
Or nearly the part - - - - -	$\frac{1}{37}$	$\frac{1}{41}$

57. *Tuesday, August* 12, 1783; *from* 10 *till* 2½.
The weather variable. Sometimes flying showers.
Barometer 30·0; Thermometer 64° at 3 P. M.

No	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	in.	lb	inches		feet
1	2			2·55							
2	2			2·50							
3	2			2·50							
4	16			24·6							
5	16			21·8							
6	16			24·5							
7	16	16 12½	1·96	36 0	19·6	88·3	8	663·0	77·35	40·20	1411
8	16	16 12½	1·96	36·7	19·8	88·6	10	664·6	77·38	40·19	1424
9	2			2·5							
10	16			28·25							
11	16			26·4							
12	16			24·7							
13	16	16 12½	1·96	39·1	23·2	87·8	11	666·3	77·41	40·19	1689
14	16	16 12½	1·96	35·8	21·7	88·5	10	667·9	77·44	40·19	1572
15	16	16 12½	1·96	37·9	23·3	91·1	11	669·6	77·47	40·19	1644
16	16	16 12½	1·96	40·7	24·8	90·6	11	671·2	77·50	40·19	1765
17	16	16 12½	1·96	42·4	24·2	91·3	10	672·9	77·53	40·18	1714

The Gun was n° 1 in the first 8 rounds; and n° 2 in the rest to the end. The weight, &c, as before.

The Pendulum was a new block, made of sound dry elm, painted, and hung in the same frame as the former; but turned end-ways, or the end of the fibres towards the gun; whereas the former was side-ways. It was firmly bound round with strong iron bars; but neither plates of iron nor lead were put within it. The dimensions of the block are,

Length from front to back -	26¾ inches
Depth of the face - - -	24¾
Breadth of the same - - -	18¼
Its weight with iron - - -	664 lb
Radius to tape as before	- 117·8 inches
To centre of gravity - -	77·35
Oscillations per minute	- 40·20

At the 7th and 15th rounds the balls struck both in firm and solid wood, when their penetrations, to the hinder part of the ball, measured $10\frac{1}{2}$ and 11 inches; so that the fore part penetrated $12\frac{1}{2}$ inches in the first case, and 13 inches in the latter.

	Gun 1	Gun 2
Mean length of the charge -	11·4	11·3
Mean recoil with ball - -	36·35	40·03 omitting nº 14
Ditto without - - - - -	23·63	26·45
Difference, or $c =$ - - -	12·72	13·58
Hence velocity by the recoil	1399	1497
Mean ditto by the pendulum	1419	1676
Difference, the recoil less -	20	179
Or nearly the part - - -	$\frac{1}{71}$	$\frac{1}{9}$

58. N. B. In this day's experiments, and those that follow, as long as the same block of wood is used, the theorems for correcting the place of the centre of gravity, and the number of oscillations per minute, as laid down at Art. 44, will be a little altered, when the weight of the pendulum is varied at the centre of the block. The reason of which is, that now the distance to the centre is 88·7, which before was only 88·3. And by using 88·7 for 88·3 in the theorems in that article, those theorems will become

$\text{G} = 88{\cdot}7 - \frac{7524}{p}$ for the new value of g, and

$\text{N} = 39{\cdot}646 + \frac{314}{p-93}$ for the new value of n.

Had i been $= 89{\cdot}3$, the new value of g and n would have been

$$\text{G} = 89{\cdot}3 - \frac{7920}{p}, \text{ and}$$

$$\text{N} = 39{\cdot}51 + \frac{386}{p-106}.$$

And these last are the proper theorems for this day's experiments, the mean distance of the points struck being nearly 89·3.

59. *Wednesday, August* 13, 1783; *from* 10 *till* 2.
The weather cloudy and misty, but it did not rain.
Barometer 30·17; Thermometer 64° at 5 P. M.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	inc	lb	inches		feet
1	2			2·5							
2	2			2·6							
3	16			27·6							
4	16			27·9							
5	16	16 12½	1·96	41·8	26·8	87·0	12	672 9	77·53	40·18	1992
6	16	16 12½	1·96	36·3	23·1	86·2	13	674·5	77·55	40·18	D
7	16	16 12½	1·96	42·3			11				
8	16	16 12½	1·96	41·0	25·2	84·7	11	677·8	77·58	40·18	1940
9	8			14·2							
10	8			13·5							
11	8			13·6							
12	8	16 12½	1·96	27·6	22·4	84·4	10	679·4	77·60	40·18	1735
13	8	16 12½	1·96	28·8	25·8	90·3		681·0	77·63	40·18	1872

The GUN was n° 3. In the 5, 6, 7, 8, and 12th rounds, the gun had from 15′ to 20′ elevation. At the 6th round an uncommon large quantity of powder came out unfired, so as to scatter a great way over the ground, and bespatter the face of the screen and pendulum very much; which was not the case in any other round. And this may account for the smaller arcs described at that number.

The PENDULUM was in the same condition as it had been left hanging after the last day's experiments, with all the balls and plugs in it. After this day's experiments, its weight was found to be 681 lb, including all the balls and plugs, except one which flew out behind the pendulum at the 7th round, occasioned by this ball striking in the same hole as n° 6, and driving it out. This ball, which came out, was quite whole and perfect; it was black on the hinder part with the powder, but rubbed bright in front with the friction in passing through the wood. The tape of the pendulum also broke at this round, so that the vibration could not be measured.

The value of i, or the mean among the distances of the point struck this day and the last is 88.

Of the plugs, this day and the last, 10 inches weighed 9 oz.

	8 oz	16 oz
Mean length of the charge -	6·0	11·1
Mean recoil with ball - -	28·2	41·7
Ditto without - - - - - -	13·77	27·75
Difference, or $c =$ - - -	14·43	13·95
Hence velocity by the recoil	1594	1542
Mean ditto by the pendulum	1803	1966
Difference, the recoil less -	209	424
Or nearly the part - - -	$\frac{1}{9}$	$\frac{1}{5}$

60. *Monday, September* 8, 1783; *from* 10 *till* 1½ P. M.

Weather windy and cloudy, with some drops of rain.

Barometer 30·03; and Thermometer 61° at 10 A. M.

N°	Powder	Ball's wt		Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Value of p	Value of g	Value of n	Veloc. ball
	oz	oz	dr	inches	inches	inches	inches	in	lb	inches		feet
1	2				2·7							
2	2				2·55							
3	2				2·6							
4	4				6·55							
5	4				6·1							
6	4				6·8							
7	4	16	13	1·96	17·4	17·8	88·1	10	663·0	77·35	40·20	1281
8	4	16	13	1·96	18·3	19·0	88·3	9	664·7	77·37	40·19	1369
9	4	16	13	1·96	18·2	18·8	88·0	8	666·3	77·40	40·19	1363
10	4	16	13	1·96	17·9	18·0	87·2	9	667·8	77·42	40·19	1321
11	2	16	13	1·96	10·9	12·4	87·8	8	669·4	77·44	40·19	906
12	2	16	13	1·96	11·2	12·5	85·8	7	670·9	77·47	40·19	937
13	2	16	13	1·96	11·0	12·5	86·1	7	672·4	77·49	40·19	936

The Gun n° 3, with every circumstance as usual; except that in the last four rounds it had 15′ elevation.

The Pendulum had been repaired, the balls and plugs taken out, a square hole cut quite through, and a sound piece fitted in; and the face covered with sheet lead as before.

Its weight at the beginning	663 lb
To the centre of gravity -	77·35 inches
To the tape - - - - -	117·8

The vibration at n° 8 a little doubtful, as the tape broke.

The plugs weighed 1 oz per inch.

The value of i, or the mean distance of the points struck, 87·3.

Weight of Powder - - - -	2 oz	4 oz
Mean length of the charge -	1·9	3·2
Mean recoil with ball - -	11·03	17·95
Ditto without - - - - - -	2·62	6·48
Difference, or c = - - - -	8·41	11·47
Hence velocity by the recoil	928	1266
Mean ditto by the pendulum	926	1334
Difference, recoil less - -	−2	68
Or nearly the part - - -	$\frac{1}{463}$	$\frac{1}{19}$

61. *Wednesday, September* 10, 1783; *from* 10 *till* 12.
The weather was fine.
Barometer 29·7; Thermometer 60° at 10 A. M.

N°	Pow-der	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	inc	lb	inches		feet
1	2			2·5							
2	2			2·2							
3	2			2·45							
4	4			5·8							
5	4			5·8							
6	4			5·7							
7	8			12·1							
8	8			12·1							
9	8			12·2							
10	8	16 12½	1·96	24·5	18·0	88·3	5	671·4	77·48	40·19	1315
11	8	16 12½	1·96	25·1	19·3	89·5	8	672·8	77·50	40·19	1394
12	8	16 12½	1·96	24·8	18·1	86·8	6	674·3	77·52	40·19	1351
13	4	16 12½	1·96	15·85	14·65	88·5	7	675·7	77·54	40·19	1075
14	4	16 12½	1·96	15·7	14·1	87·4	6	677·2	77·56	40·19	1050
15	4	16 12½	1·96	16·35	15·4	88·5	3	678·6	77·59	40·18	1136
16	2	16 12½	1·96	10·0	10·75	89·3	4	679·8	77·61	40·18	787
17	2	16 12½	1·96	9·9	10·6	89·8	3	681·1	77·63	40·18	774
18	2	16 12½	1·96	10·1	10·65	88·0	3	682·3	77·65	40·18	795

The GUN n° 1. Weight and other circumstances as usual.

The PENDULUM as left hanging since Monday. Its radius, &c, as usual.—The value of i, or the mean distance among the points struck this day and the former, is 88·0.

The plugs weighed 1 oz per inch.

Weight of Powder	2 oz	4 oz	8 oz
Mean length of the charge -	1·9	3·2	5·7
Mean recoil with ball - -	10·00	15·97	24·8
Ditto without - - - - -	2·38	5·77	12·1
Difference, or c = - - - -	7·62	10·20	12·7
Hence velocity by the recoil	838	1122	1396
Mean ditto by the pendulum	785	1087	1353
Difference, the recoil more,	53	35	43
Or nearly the part - -	$\frac{1}{15}$	$\frac{1}{31}$	$\frac{1}{31}$

62. *Thursday, September* 11, 1783 ; *from* 10 *till* 12.

The weather was fine.

Barometer 29·93 ; Thermometer 60° at 10 A. M.

N°	Powder	Ball's wt	Ball's diam	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	oz dr	inches	inches	inches	inches	inc	lb	inches		feet
1	2			2·65							
2	2			2·7							
3	2			2·65							
4	4			6·2							
5	4			6·1							
6	4			6·0							
7	8			13·7							
8	8			13·1							
9	8			14·1							
10	8	16 12½	1·96	27·0	21·2	87·4	4	681·2	77·63	40·18	1590
11	8	16 12½	1·96	27·1	21·3	88·1	6	682·5	77·65	40·18	1589
12	8	16 12½	1·96	26·3	20·2	86·7	12	683·9	77·67	40·18	1535
13	4	16 12½	1·96	17·3	16·6	87·5	9	685·7	77·70	40·18	1253
14	4	16 12½	1·96	17·1	16·7	89·9	8	687·3	77·72	40·18	1230
15	4	16 12½	1·96	17·1	16·7	89·9	7	688·8	77·75	40·17	1233
16	2	16 12½	1·96	10·3	11·5	90·1	4	690·4	77·77	40·17	849
17	2	16 12½	1·96	10·45	11·7	90·3	3	691·7	77·80	40·17	864
18	2	16 12½	1·96	10·3	11·5	89·9	2	692·9	77·82	40·17	855

The Gun n° 2. In the last 5 rounds it had about 10 depression.

The Pendulum the same as left hanging since yesterday. After the experiments were concluded to-day, it weighed 694 lb.—The plugs weighed 1 oz per inch.

The weight of balls and plugs lodged in the block, these last three days, was 36 lb ; which added to 663, the weight at the beginning, makes 699 : but it weighed at the end only 694 ; so that it lost 5 lb of its weight in the 4 days, or 1¼ lb per day on a medium.

The value of i, or the mean among the distance of the points struck these three days, is 88·3.

	2 oz	4 oz	8 oz
Mean length of the charge -	1·8	3·1	5·7
Mean recoil with ball - -	10·35	17·17	26·80
Ditto without - - - - - -	2·67	6·10	13·63
Difference or c = - - - -	7·68	11·07	13·17
Hence velocity by the recoil	846	1220	1452
Mean ditto by the pendulum	856	1239	1571
Difference, the gun less -	10	19	119
Or nearly the part - - -	$\frac{1}{86}$	$\frac{1}{65}$	$\frac{1}{13}$

63. *Tuesday, September* 16, 1783 ; *from* 12 *till* 2.

The weather was rainy.—Gun n° 1.

Barometer 29·9 ; Thermometer 64° at noon.

N°	Powder			Vibration of gun							
	oz										
1	2			2·3							
2	2			2·5							
3	2			2·35							
4	4			5·25							
5	4			5·05							
6	4			5·4							
7	8			11·65							
8	8			11·9							
9	8			12·05							
10	12			17·3							
11	12			19·3							
12	12			18·7							
13	12			17·1							
14	16			25·3							
15	16			23·3							
16	16			24·0							
17	20			28·5							
18	20			28·2							
19	20			24·8							

The last n° very uncertain ; the tape entangled.

	2 oz	4 oz	8 oz	12 oz	16 oz	20 oz
Mean length of charge - -	1·9	3·2	5·6	8·2	10·6	13·2
Mean recoil, omitting n° 19,	2·38	5·23	11·9	18·1	24·2	28·2

64. *Thursday, September* 18, 1783 ; *from* 10 *till* 3 P. M.

The weather fair and mild.

Barometer 30·08 ; Thermometer 64° at 10 A. M.

N°	Powder wt	Powder ht	Ball's wt	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Velocity of the ball
	oz	inches	oz dr	inches	inches	inches		lbs	inches		feet
1											
2											
3											
4											
5											
6											
7	24	14·5	16 12½	38·6	17·3	90·8	14	655·0	77·21	40·21	1194
8	32	21·6	16 12½	44·0	13·0	92·9	8	656·8	77·24	40·20	880
9	36	24·4	16 12½	45·8	12·3	92·5	7	658·3	77·27	40·20	838
10	39	27·2	16 12½	47·5	.	went	over	659·7	77·29	40·20	
11	20	13·3	16 12½	36·7	15·5	85·8	11	660·8	77·29	40·20	1144
12	12	8·1	16 12½	29·9	18·75	86·8	11	662·5	77·32	40·20	1371
13	14	9·3	16 12½	27·3	16·2	85·8	11	664·2	77·35	40·20	1202 D
14	10	6·9	16 13	28·5	19·15	86·6	10	665·9	77·37	40·20	1409
15	14	10·1	16 13	31·2	19·0	89·5	10	667·5	77·40	40·19	1357
16	16	11·1	16 13	32·7	18·0	91·4	5	669·1	77·43	40·19	1262 D
17	8	5·7	16 13	26·4	20·05	89·7	9	670·4	77·45	40·19	1436
18	6	4·6	16 13	20·7	17·0	88·5	9	671·9	77·48	40·19	1237
19	12	8·4	16 13	29·0	18·55	89·9	7	673·4	77·51	40·19	1333
20	10	6·9	16 13	28·8	20·05	90·5	6	674·8	77·53	40·19	1434
21	14	9·6	16 13	32·2	18·5	91·1	6	676·2	77·56	40·19	1318
22	8	5·5	16 13	25·3	18·8	89·9	6	677·6	77·59	40·18	1360
23	12	8·4	16 13	31·1	19·8	90·9	6	678·9	77·61	40·18	1420
24	10	7·0	16 13	28·6	19·35	89·7	6	680·2	77·64	40·18	1409
25	8	5·7	16 13	25·4	18·5	89·5	5	681·5	77·67	40·18	1354
26	16	10·7	16 13	31·2	16·75	89·3	5	682·8	77·69	40·18	1231 D
27	16	11·1	16 13	31·5	17·2	91·4	5	684·1	77·72	40·18	1238 D
28	14	9·7	16 13	33·4	20·0	90·5	4	685·4	77·75	40·17	1457

The GUN n° 1. The charge of powder was gradually increased till the gun became quite full at n° 10, when there was just room for half the ball to lie within the muzzle;

which being too short a length to give direction to the ball, it missed the pendulum, going over and just striking the top of the screen frame, about $21\frac{1}{2}$ inches above the line of direction, which, though a very slender piece of wood, turned the ball up into a still higher direction, in which it struck the bank over the pendulum, and entered it sloping, though but a little way: all which circumstances show that the force of the ball was but small. And even at the 9th round, when the centre of the ball was about 3 inches within the gun, the ball struck the pendulum 5 inches out of the line of direction. The gun was scarce ever sensibly heated.

The diameter of the balls 1·96 inches.

The PENDULUM had been gutted, and had received a new core. It was hung up in the morning of the day before yesterday, when it weighed 659 lb. And when taken down this evening it weighed only 686 lb, which is near 4 lb less than the balls and plugs ought to make it; and which 4 pounds must have evaporated in the 3 days.

The plugs weighed $\frac{7}{8}$ of an ounce to the inch.

The value of i, or mean point struck, 89·7.

All the three rounds with 16 oz are very doubtful, and seem to be too low, from some unknown cause.

Mean velocity by the pendulum, &c.

Powder wt	Powder ht	Recoil gun	Veloc. ball
8	5·6	25·7	1383
10	6·9	28·6	1417
12	8·3	30·0	1375
14	9·7	32·3	1333
16	11·0	31·8 D	1243 D
20	13·3	36·7	
24	14·5	38·6	
32	21·6	44·0	
36	24·4	45·8	
39	27·2	47·5	

65. *Thursday, September* 25, 1783; *from* 10 A. M. *till* 3 P. M.

Fine, clear, and warm weather.

Barometer 29·93; Thermometer 59° at 10 A. M.

N°	Powder wt	Powder ht	Ball's wt		Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball	
	oz	inches	oz	dr	inches	inches	inches	inc	lb	inches		feet	
1	8	5·9	16	13	27·6	23·8	89·3	14	643·0	77·00	40·22	1632	
2	10	7·2	16	13	29·5	23·0	88·7	15	644·9	77·03	40·22	1593	
3	12	8·4	16	12½	32·0	22·0	88·7	15	646·8	77·06	40·21	1532	
4	14	9·4	16	12½	32·2	21·3	88·5	15	648·8	77·09	40·21	1491	D
5	16	11·3	16	12½	39·4	23·6	88·3	13	650·8	77·12	40·21	1662	
6	18	12·3	16	12½	37·1	21·0	89·0	12	652·6	77·16	40·21	1472	
7	20	13·2	16	12	39·8	21·9	91·6	17	654·4	77·19	40·21	1499	
8	22	15·1	16	12	41·5	21·7	91·5	11	656·5	77·22	40·20	1492	D
9	24	15·8	16	12	42·6	21·3	91·6	10	658·2	77·25	40·20	1468	
10	28	18·9	16	12	44·2	18·1	90·5	10	659·8	77·28	40·20	1266	D
11	32	22·1	16	12	52·8	20·3	90·8	9	661·5	77·31	40·20	1419	
12	8	5·5	16	12	27·0	22·2	90·5	8	663·1	77·35	40·20	1562	
13	10	7·0	16	12	30·5	22·9	90·0	7	664·6	77·38	40·20	1624	
14	12	8·1	16	12	32·4	21·8	85·2	15	666·1	77·41	40·19	1638	
15	14	9·3	16	12	32·9	20·4	86·4	15	668·0	77·44	40·19	1517	
16	16	10·9	16	12	39·0	22·2	85·8	13	670·0	77·47	40·19	1667	
17	8	5·5	16	12	25·2	19·6	87·9	15	671·8	77·50	40·19	1441	D
18	8	5·5	16	12	26·3	20·8	89·2	13	673·6	77·53	40·19	1512	
19	10	6·7	16	12	26·3	19·6	89·1	13	675·4	77·56	40·18	1431	D
20	12	8·2	16	12	32·7	22·8	88·8	11	677·2	77·59	40·18	1675	
21	14	9·1	16	12	35·1	17·7	89·0	11	678·9	77·63	40·18	1302	D
22	16	10·4	16	12	32·8	15·6	89·1	8	680·6	77·66	40·18	1149	D
23	6	4·4	16	12	22·8	14·5	89·1		682·1	77·69	40·18	1070	D

The GUN was n° 2.

The diameter of the balls 1·96 inches.

The PENDULUM had been repaired with a new core, but of very soft and damp wood. It was hung up yesterday morning, when it weighed 653 lb. And when taken down this evening it weighed only 678 lb with all the balls and

plugs, the whole ball which came out behind, as well as the broken pieces of the wood and balls which flew out in the latter rounds, being collected and weighed with it; which is about 15½ lb less than it ought to be; so that about 15½ lb has been lost by evaporation in the space of 30 hours, or about half a pound an hour.

At n^os 4, 8, 10 the tape of the pendulum entangled and broke, which rendered those vibrations doubtful, as marked D. Some other rounds are marked doubtful, from some other cause, perhaps the badness of the wood in the pendulum, which split very much; from which circumstance part of the force of the ball might be lost by the lateral pressure.

The plugs weighed 14 oz to 15 inches.

The value of i, or the mean point struck, 89·5 inches.

The penetration at the 1st and 7th rounds, which were made in fresh parts of the wood, were from 19 to 20 inches; so that the fore part of the ball penetrated about 21½ inches in this soft wood.

Mean recoil and velocity by the pendulum.

Powder	Recoil	Veloc.
8	26·7	1569
10	30·0	1608
12	32·4	1615
14	33·4 D	1517 D
16	36·8 D	1664 D
18	37·1	
20	39·8	
22	41·5	
24	42·6	
28	44·2	
32	52·8 D	

But these mediums are not much to be depended on, as the velocities are all very irregular. It is, in particular, highly probable that the velocity here found for 14 oz of powder is too small, and that for 16 oz too great.

66. *Monday, September* 29, 1783; *from* 10 A.M. *till* $1\frac{1}{2}$ P.M.
The weather fine, clear, and warm.
Barometer 30·28; Thermometer 64° at 10 A.M.

N°	Powder		Ball's wt.		Vibration of		Point struck	Plugs	Values of			Veloc. of the ball	
	wt	ht			gun	pend			p	g	n		
	oz	inches	oz	dr	inches	inches	inches	in	lb	inches		feet	
1	2				2·8								
2	2				2·75								
3	6	4·5	16	$11\frac{1}{2}$	22·6	20·5	88·9	7	654·0	77·20	40·21	1448	
4	8	5·6	16	$11\frac{1}{2}$	26·9	22·1	89·1	10	655·4	77·22	40·21	1561	
5	10	6·9	16	$11\frac{1}{2}$	30·2	23·4	91·3	8	656·9	77·25	40·20	1618	
6	12	8·3	16	$11\frac{1}{2}$	33·3	23·9	90·6	9	658·3	77·27	40·20	1669	
7	14	9·5	16	$11\frac{1}{2}$	37·4	24·7	88·9	10	659·8	77·30	40·20	1763	
8	16	10·7	16	$11\frac{1}{2}$	35·9	21·5	87·3	9	661·3	77·32	40·20	1566	D
9	16	11·0	16	$11\frac{1}{2}$	40·1	23·5	87·8	8	662·8	77·35	40·20	1707	
10	18	12·1	16	$11\frac{1}{2}$	32·7	18·3	87·1	6	664·2	77·37	40·20	1343	D
11	18	12·2	16	$11\frac{1}{2}$	39·5	21·9	87·7	12	665·5	77·40	40·20	1598	
12	20	13·0	16	11	42·7	23·4	91·8	10	667·1	77·43	40·19	1639	
13	14	9·6	16	11		21·6	89·3	9	668·6	77·46	40·19	1561	

The GUN n° 2.—At the last round the tape broke, so that the recoil could not be measured. N^os^ 8 and 10 are plainly both irregular, the recoils being greatly deficient: the vibrations of the pendulum might perhaps be defective by the balls being resisted sideways by the wood, or by changing their direction within the block; but there is no cause which I can suspect for the defective recoils of the gun, as all the circumstances were alike in every case, and the heights of the charges show that there was no mistake in the quantity of powder.—At the last firing the vent had a small channel blown in it, though the gun was no where very hot.

The PENDULUM had received a new core of sound dry elm, and weighed this morning, when it was hung up, 654 lb.

The diameter of the balls 1·96 inches.

The plugs weighed $6\frac{1}{4}$ oz to 8 inches.

The value of i, or mean point struck, 89·1.

The first penetration was 12 inches measured behind the ball, and consequently the fore part penetrated 14 inches.

Mean recoil of gun and velocity of ball:

Powder	Recoil	Veloc.
6	22·6	1448
8	26·9	1561
10	30·2	1618
12	33·3	1669
14	37·4	1662
16	38·0	1637
18	39·5	1598
20	42·7	1639

67. *Thursday, September* 30, 1783; *from* 10 P.M. *till* 1½ A.M.

Fine, clear, and warm weather.

Barometer 30·25; Thermometer 64° at 10 A. M.

Nº	Powder wt	Powder ht	Ball's wt	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	inch	oz dr	inches	inches	inches	inc	lb	inches		feet
1	2	1·9		2·4							
2	2	1·9		2·6							
3	10	7·1	16 14	27·5	19·4	89·6	6	669·0	77·46	40·19	1383
4	12	8·4	16 14	31·9	20·4	88·7	6	670·3	77·48	40·19	1472
5	8	5·8	16 14	25·3	18·7	88·8	6	671·6	77·51	40·19	1351
6	14	9·3	16 14	32·7	19·4	88·9	7	672·9	77·53	40·19	1403
7	6	4·6	16 14	21·6	18·5	90·3	6	674·3	77·55	40·19	1320
8	10	6·9	16 14	27·5	19·8	91·2	7	675·6	77·58	40·19	1403
9	12	8·3	16 14	29·5	19·9	92·2	5	677·0	77·60	40·18	1398
10	8	5·7	16 14	25·3	18·9	90·2	6	678·3	77·62	40·18	1360
11	14	9·6	16 14	32·6	19·7	91·2	6	679·6	77·65	40·18	1405
12	6	4·4	16 14	21·9	18·0	87·0	7	680·9	77·67	40·18	1349
13	10	6·9	16 14	28·7	18·8	86·5	7	682·3	77·69	40·18	1420
14	12	8·4	16 14	31·7	20·0	87·9	6	683·7	77·71	40·18	1490
15	8	6·0	16 14	26·4	19·4	88·0	7	685·0	77·74	40·17	1447
16	14	9·3	16 14	32·2	18·4	86·5	7	686·4	77·76	40·17	1399
17	6	4·3	16 14	21·7	17·9	89·2	5	687·8	77·78	40·17	1323

The GUN nº 1.—The vent blew a little, though the gun was never very warm.

The PENDULUM was the same as it hung since yesterday with all the balls in it; but the other end of it was turned, which bore the firings very well, the core being of sound dry wood. At the end of the experiments this day the pendulum weighed 689 lb, which is only 1 lb less than it ought to be by the addition of the balls and plugs to the first weight; so little was its loss of weight by evaporation, owing to the dryness of the wood.

The diameter of the balls 1·96 inches.

The plugs weighed 6¼ oz to 8 inches.

The value of i, or mean point struck, 89·1 inches.

The first penetration, being in sound wood, was 14¼ inches to the fore part of the ball.

This set of experiments, as well as those of the three preceding days, were made to determine the best charge, or that which gives the greatest velocity.

This is a good set of experiments.

Mean recoil, and velocity of the ball by the pendulum, are as follow,

Powder	Recoil	Veloc.
6	21·7	1331
8	25·6	1386
10	27·9	1402
12	31·0	1453
14	32·5	1402

which velocities, as well as the recoils, are found by adding those of each sort together, and dividing by the number of them, as below:

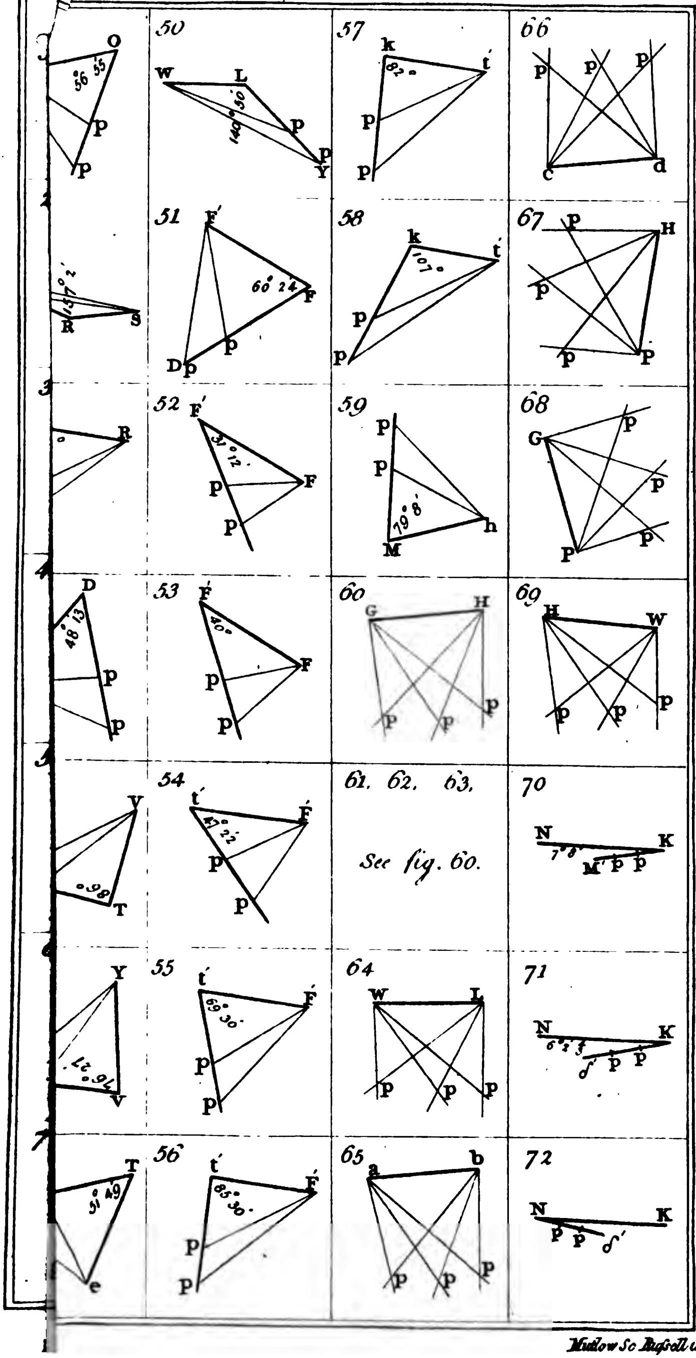
	6	8	10	12	14
	1320	1351	1382	1472	1403
	1349	1360	1403	1398	1405
	1323	1447	1420	1490	1399
3)	3992	4158	4206	4360	4207
means	1331	1386	1402	1453	1402

where the velocity with 12 oz is greatest.

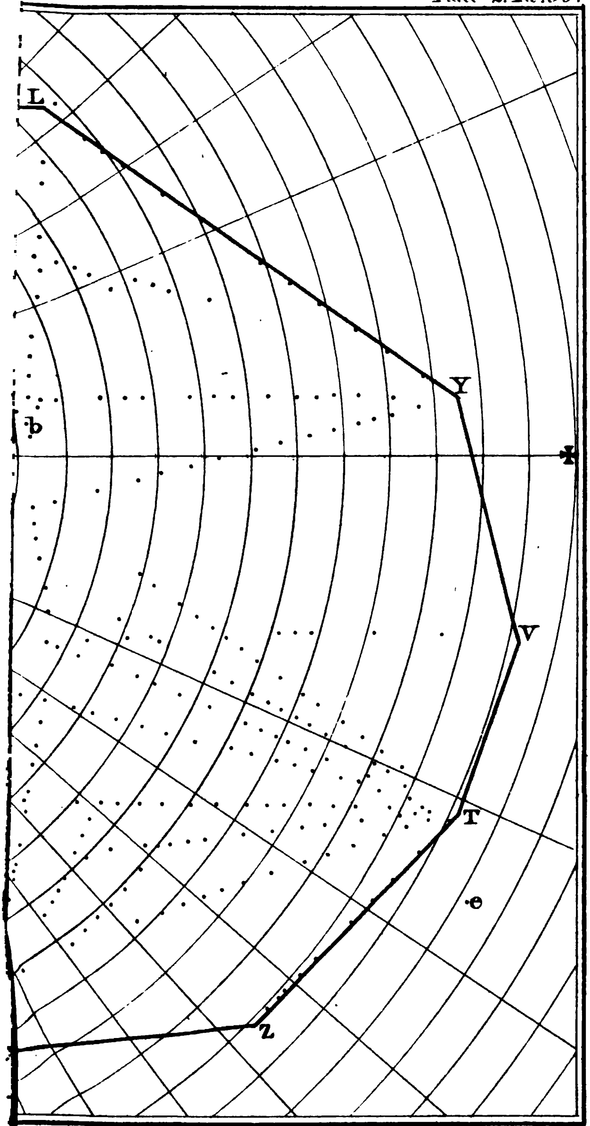
END OF VOL. II.

T. DAVISON, Lombard-street,
Whitefriars, London.

Plate 1. Pa. 68.



Mutlow Sc Russell st



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1.

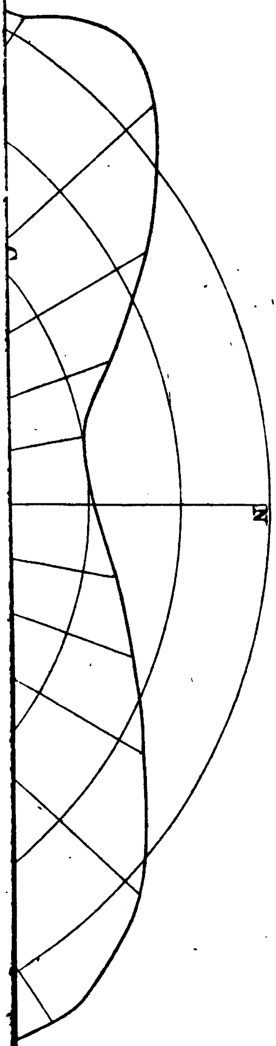


Plate 3. Pa. 68.

Fig. 2.

A.F. or Base.	Perpen-dicular.	Diff.
100	100	·001
106	106	·001
112	112	·001
119	119	·001
126	126	·001
133	133	·001
141	141	·001
150	150	·002
159	159	·002
168	168	·002
178	178	·003
188	188	·003
199	199	·004
211	211	·005
224	224	·005
237	237	·006
251	251	·008
266	266	·009
282	282	·011
299	299	·013
316	316	·015
335	335	·017
355	355	·020
376	376	·024
398	398	·028
422	422	·033
447	447	·039
473	473	·045
501	501	·053
531	531	·062
562	562	·072
596	596	·084
631	631	·097
668	668	·113
708	708	·130
750	750	·150
794	794	·172
842	842	·198
891	891	·226
944	944	·258
1000	1000	·293

Plate 4. Pa. 68.

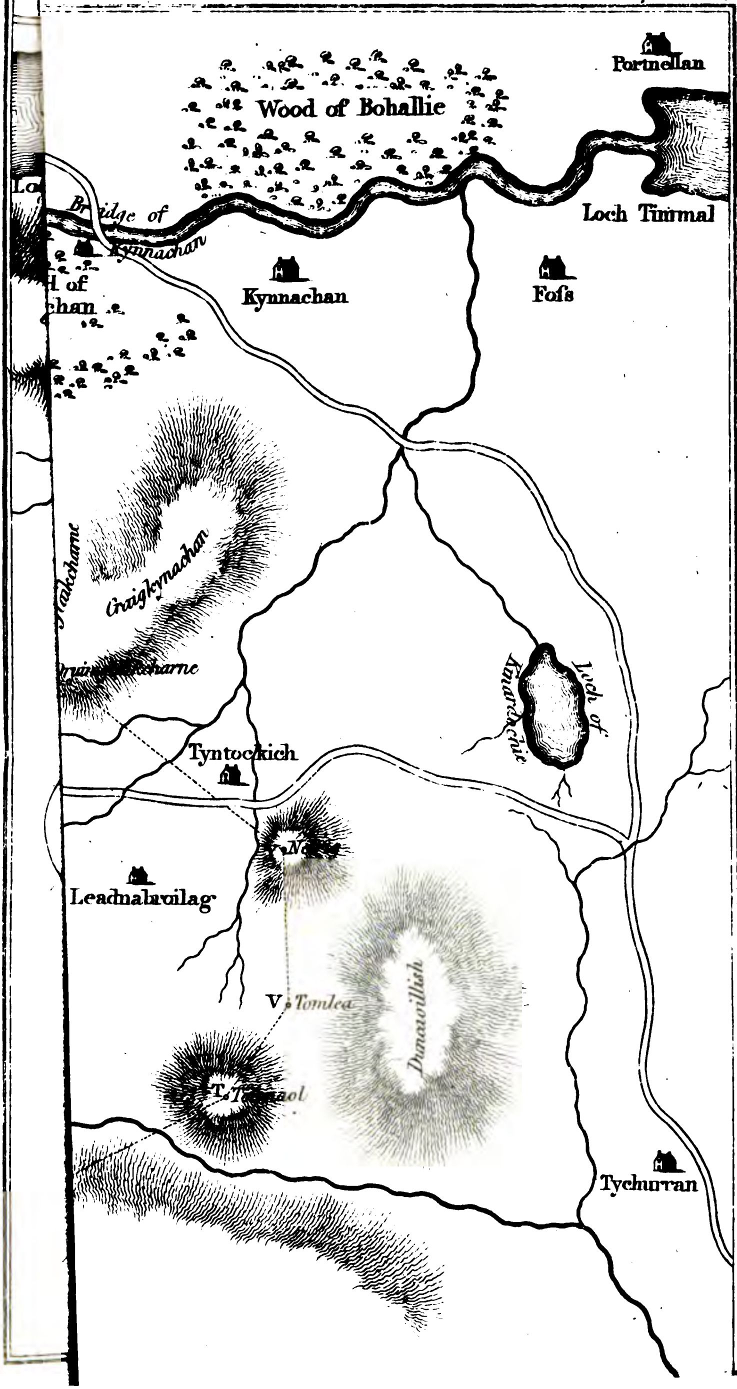
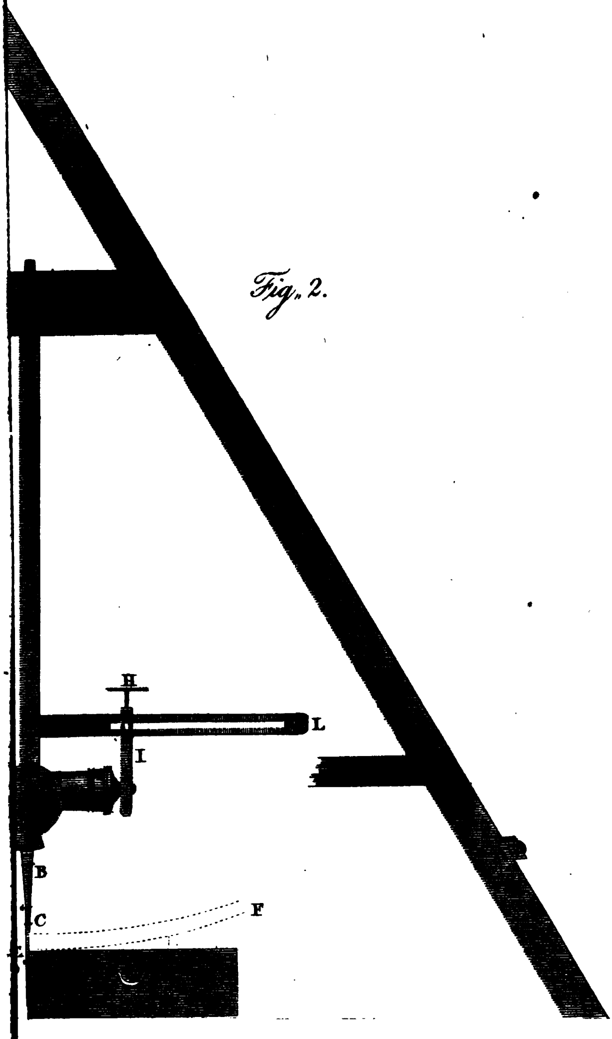

Portnellan
Wood of Bohallie
Loch Tummal
Bridge of Kynnachan
Kynnachan
Foss
Craigkynachan
Loch of Kinardochie
Tyntockich
V. Tomlea
Tychurran

Plate V. Pa. 384.

Fig. 2.



Mutlow Sc. Rufsell Ct.